Resolving New Keynesian Anomalies with Wealth in the Utility Function

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At the zero lower bound, the New Keynesian model predicts that output and inflation collapse to implausibly low levels, and that government spending and forward guidance have implausibly large effects. To resolve these anomalies, we introduce wealth into the utility function; the justification is that wealth is a marker of social status, and people value social status. Since people save not only for future consumption but also to accrue social status, the Euler equation is modified. As a result, when the marginal utility of wealth is sufficiently large, the dynamical system representing the equilibrium at the zero lower bound transforms from a saddle to a source—which resolves all the anomalies.
1. Introduction

A current issue in monetary economics is that the New Keynesian model makes several anomalous predictions when the zero lower bound on nominal interest rates (ZLB) is binding: (1) collapse of output and inflation to implausibly low levels (Egbertsson and Woodford 2004; Eggertsson 2011; Werning 2011); (2) implausibly large effect of forward guidance (Del Negro, Giannoni, and Patterson 2015; Carlstrom, Fuerst, and Paustian 2015; Cochrane 2017); and (3) implausibly large effect of government spending (Christiano, Eichenbaum, and Rebelo 2011; Woodford 2011; Cochrane 2017).

Several papers have developed variants of the New Keynesian model that behave well at the ZLB, and so are not plagued by any of the anomalies (Gabaix 2016; Diba and Loisel 2019; Cochrane 2018; Bilbiie 2019; Acharya and Dogra 2019). But these variants are more complex than the standard model, either because the derivations are more complicated (due to bounded rationality or heterogeneity), or because the dynamical system describing the equilibrium is higher-dimensional (due to heterogeneity, bank-reserve dynamics, or a fiscal theory of the price level). Moreover, a good chunk of the description and resolution of the anomalies is conducted by numerical simulations. In consequence, it is sometimes difficult to grasp the nature of the anomalies and the mechanism behind their resolutions.

It may therefore be valuable to strip the logic to the bone. We do so using a New Keynesian model in which relative wealth enters the utility function. The justification for the assumption is that relative wealth is a marker of social status, and people value high social status. The deviation from the standard model is minimum: the derivations are the same, and the equilibrium system remains 2-dimensional—there is only an extra term in the Euler equation. Furthermore, we veer away from numerical simulations and establish our results with phase diagrams describing the equilibrium dynamics of output and inflation, as in Werning (2011). The dynamics are governed by two differential equations: the Phillips curve, describing how prices are set, and the Euler equation, describing how consumers save. The model’s simplicity and the phase diagrams will allow us to gain new insights into the anomalies and their resolutions.¹

Using the phase diagrams, we begin by depicting the anomalies. First, we find that output

¹Our approach relates to the work of Michaillat and Saez (2014), Ono and Yamada (2018), and Michau (2018). By assuming wealth in the utility function, they obtain non-New-Keynesian models that behave well at the ZLB. But their results are not portable to the New Keynesian framework because they require strong forms of wage or price rigidity (exogenous wages, fixed inflation, or downward nominal wage rigidity). Our approach also relates to the work of Fisher (2015) and Campbell et al. (2017), who build New Keynesian models with government bonds in the utility function. The bonds-in-the-utility assumption captures special features of government bonds relative to other assets, such as safety and liquidity (for example, Krishnamurthy and Vissing-Jorgensen 2012). While their assumption and ours are conceptually different, they affect equilibrium conditions in a similar way. These papers use their assumption to generate risk-premium shocks (Fisher) and to alleviate the forward-guidance puzzle (Campbell et al.).
and inflation become unboundedly low when the ZLB episode is arbitrarily long-lasting. Second, we find that there is a duration of forward guidance above which any ZLB slump, irrespective of its duration, is transformed into a boom. Such boom is unbounded when the ZLB episode is arbitrarily long-lasting. Third, we find that there is an amount of government spending at which the government-spending multiplier becomes infinite when the ZLB episode lasts an arbitrarily long time. Furthermore, when government spending exceeds this amount and the ZLB episode is arbitrarily long-lasting, the economy experiences an unbounded boom at the ZLB.

The phase diagrams also pinpoint the origin of the anomalies: they arise because the Phillips curve-Euler equation system is a saddle at the ZLB. In normal times, in contrast, the system is source, so anomalies are absent. In economic terms, the anomalies arise because household consumption responds too strongly to the real interest rate. Indeed, since the only motive for saving is future consumption, households are very forward-looking, and their response to interest rates is very strong.

Once wealth enters the utility function, however, people save not only for future consumption but also because they enjoy holding wealth in itself. They are less forward-looking, so their consumption responds less to future interest rates, which resolves all the anomalies. Mathematically, with enough marginal utility of wealth, the Euler equation is sufficiently “discounted,” in the sense of McKay, Nakamura, and Steinsson (2017), that at the ZLB the Phillips curve-Euler equation system transforms from a saddle to a source. At that point the response to a temporary shock, however long-lasting, is a muted version of the response to a permanent shock, which is finite, so the anomalies disappear.

Indeed, even when the ZLB is arbitrarily long-lasting, our model makes reasonable predictions. First, output and inflation never collapse: they are bounded below by the ZLB steady state. Second, when the ZLB episode is long enough, the economy necessarily experiences a slump—irrespective of the duration of forward guidance. Third, government-spending multipliers are always finite, even when the ZLB episode lasts an arbitrarily long time.

Beside its anomalies, the New Keynesian model has several other intriguing properties at the ZLB—some labeled “paradoxes” because they defy usual economic logic (Eggertsson 2010; Werning 2011; Eggertsson and Krugman 2012). We find that our model shares these properties. First, the paradox of thrift holds: when all households desire to save more than their neighbors, the economy contracts and they end up saving the same amount as the neighbors. The paradox of toil also holds: when all households desire to work more, the economy contracts and they end up working less. The paradox of flexibility is present too: the economy contracts when prices become more flexible. Last, the government-spending multiplier is above one, so public spending stimulates private consumption.
**Justification for wealth in the utility function.** The standard model assumes that people only save to smooth consumption over time, but it has long been recognized that people save for many reasons beside future consumption. For instance, observing the behavior of the European upper class in the early 20th century, Keynes (1919, chap. 2) noted that “The duty of saving became nine-tenths of virtue and the growth of the cake the object of true religion…. Saving was for old age or for your children; but this was only in theory—the virtue of the cake was that it was never to be consumed, neither by you nor by your children after you.” A few years later, Irving Fisher observed that “A man may include in the benefits of his wealth ...the social standing he thinks it gives him, or political power and influence, or the mere miserly sense of possession, or the satisfaction in the mere process of further accumulation” (Fisher 1930, p. 17). Fisher’s perspective is particularly interesting since he is often credited for the modern theory of saving based on consumption smoothing.

Neuroscientific evidence also supports the view that wealth itself provides utility, independently of the future consumption that it can buy. Camerer, Loewenstein, and Prelec (2005, p. 32) note that “brain-scans conducted while people win or lose money suggest that money activates similar reward areas as do other ‘primary reinforcers’ like food and drugs, which implies that money confers direct utility, rather than simply being valued only for what it can buy.”

Among all the reasons why people may value wealth in itself, we focus on social status: we postulate that people enjoy wealth because it provides social status, and we therefore introduce relative (not absolute) wealth into the utility function. This assumption is convenient: in equilibrium, everybody is the same, so everybody’s relative wealth is zero. And the assumption seems plausible. Adam Smith, Ricardo, John Rae, J.S. Mill, Marshall, Veblen, and Frank Knight all supported the idea that people accumulate wealth to attain high social status (Steedman 1981). More recently, a broad literature documents that people seek to achieve high social status, and that accumulating wealth is a prevalent pathway to do so (Weiss and Fershtman 1998; Heffetz and Frank 2011; Fiske 2010; Anderson, Hildreth, and Howland 2015; Cheng and Tracy 2013; Ridgeway 2014; Mattan, Kubota, and Cloutier 2017).²

Additionally, the wealth-in-the-utility assumption has been found useful in models of long-run growth (Kurz 1968; Konrad 1992; Zou 1994; Corneo and Jeanne 1997; Futagami and Shibata 1998; Corneo and Jeanne 2001), risk attitudes (Robson 1992; Clemens 2004), asset pricing (Bakshi and Chen 1996; Gong and Zou 2002; Kamihigashi 2008; Michau, Ono, and Schlegl 2018), life-cycle consumption (Zou 1995; Carroll 2000; Francis 2009; Straub 2019), social stratification (Long and...

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²Here we assume that wealth directly confers utility. Cole, Mailath, and Postlewaite (1992, 1995) have developed models in which relative wealth does not directly confer utility but has other attributes such that in reduced form people behave as if wealth entered their utility function. For example, in one model, wealthier individuals have higher social rankings, which allows them to marry wealthier partners and then enjoy higher utility.
Shimomura 2004), international macroeconomics (Fisher 2005; Fisher and Hof 2005), financial crises (Kumhof, Ranciere, and Winant 2015), and optimal taxation (Saez and Stantcheva 2018). Such wide-ranging usefulness lends further support to the assumption.

Finally, our results require the marginal utility of wealth to be large enough. To assess this condition, we rely on the insight that when people derive utility from wealth, they are willing to save even if the real interest rate is below their time discount rate. Thus, we recover the marginal utility of wealth from the gap between time discount rate and real interest rate. Numerous studies have used experiments to directly estimate time discount rates. They usually find that time discount rates are significantly above market interest rates (Frederick, Loewenstein, and O’Donoghue 2002; Andersen et al. 2014). Hence, for a preponderance of available estimates, the marginal utility of wealth is sufficiently large.

2. Model

We extend the New Keynesian model by assuming that households derive utility not only from consumption and leisure but also from relative wealth. To simplify derivations and be able to represent the equilibrium with phase diagrams, we use an alternative formulation of the New Keynesian model, inspired by Benhabib, Schmitt-Grohe, and Uribe (2001) and Werning (2011). Our formulation features continuous time instead of discrete time; self-employed households instead of firms and households; and Rotemberg (1982) pricing instead of Calvo (1983) pricing.

2.1. Assumptions

The economy is composed of a measure 1 of self-employed households.

Production. Each household $j \in [0, 1]$ produces $y_j(t)$ units of a good $j$ at time $t$, sold to other households at a price $p_j(t)$. The household's production function is

$$y_j(t) = ah_j(t),$$

where $a > 0$ represents the level of technology, and $h_j(t)$ is hours of work. Working causes a disutility

$$\kappa h_j(t);$$

the parameter $\kappa > 0$ is the marginal disutility of labor.
Pricing. The goods produced by households are imperfect substitutes for one another, so each household exercises some monopoly power. Moreover, households face a quadratic cost when they change their price: changing a price at a rate $\pi_j(t) = \dot{p}_j(t)/p_j(t)$ causes a disutility

$$\frac{\gamma}{2} \pi_j(t)^2.$$ 

The parameter $\gamma > 0$ governs the cost to change prices and thus price rigidity.

Consumption. Each household consumes goods produced by other households. Household $j$ buys quantities $c_{jk}(t)$ of the goods $k \in [0, 1]$. These quantities are aggregated into a consumption index

$$c_j(t) = \left[ \int_0^1 c_{jk}(t)^{(\epsilon - 1)/\epsilon} \, dk \right]^{\epsilon/(\epsilon - 1)},$$

where $\epsilon > 1$ is the elasticity of substitution between goods. The consumption index yields utility $\ln(c_j(t))$. Given the consumption index, the relevant price index is

$$p(t) = \left[ \int_0^1 p_j(t)^{1-\epsilon} \, di \right]^{1/(1-\epsilon)}.$$

When households optimally allocate their consumption expenditure across goods, $p(t)$ is the price of one unit of consumption index. The inflation rate is defined as $\pi(t) = \dot{p}(t)/p(t)$.

Saving. Households save using government bonds. Since we postulate that people derive utility from their relative real wealth, and since bonds are the only store of wealth, holding bonds provides direct utility. Formally, holding a nominal quantity of bonds $b_j(t)$ yields utility

$$u \left( \frac{b_j(t)}{p(t)} - \frac{b(t)}{p(t)} \right).$$

The function $u : \mathbb{R} \to \mathbb{R}$ is increasing and concave, $b(t) = \int_0^1 b_k(t) \, dk$ is average nominal wealth, and $[b_j(t) - b(t)]/p(t)$ is the household $j$'s relative real wealth.

Bonds earn a nominal interest rate $i(t)$, determined by monetary policy. The law of motion of bond holdings is

$$\dot{b}_j(t) = i(t)b_j(t) + p_j(t)y_j(t) - \int_0^1 p_k(t)c_{jk}(t) \, dk - \tau(t).$$
The term $i(t)b_j(t)$ is interest income; $p_j(t)y_j(t)$ is labor income; $\int_0^1 p_k(t)c_{jk}(t)\,dk$ is consumption expenditure; and $\tau(t)$ is a lump-sum tax (used among other things to service government debt).

2.2. Household problem

The problem of household $j$ is to choose time paths for $y_j(t)$, $p_j(t)$, $h_j(t)$, $\pi_j(t)$, $c_{jk}(t)$ for all $k \in [0,1]$, and $b_j(t)$ to maximize the discounted sum of instantaneous utilities

$$\int_0^\infty e^{-\delta t} \left[ \ln(c_j(t)) + u\left(\frac{b_j(t)}{p(t)} - \frac{b(t)}{p(t)}\right) - \kappa h_j(t) - \frac{\gamma}{2} \pi_j(t)^2 \right] dt,$$

where $\delta > 0$ is the time discount rate.

The household faces four constraints: the production function (1); the budget constraint (4); the law of motion $\dot{\pi}_j(t) = \pi_j(t)p_j(t)$; and the demand for good $j$ coming from other households’ maximization:

$$y_j(t) = \left[\frac{p_j(t)}{p(t)}\right]^{-\epsilon} c(t),$$

where $c(t) = \int_0^1 c_k(t)\,dk$ is aggregate consumption. The household also faces a borrowing constraint preventing Ponzi schemes.

The household takes as given aggregate variables, initial wealth $b_j(0)$, and initial price $p_j(0)$.

2.3. Equilibrium

We assume that all households face the same initial conditions, so they all behave the same. The equilibrium can be described by two differential equations: a Phillips curve and an Euler equation, determining inflation $\pi(t)$ and aggregate output $y(t)$. We present the two equations here; heuristic derivations are in the next section; formal derivations are in appendix A; and a discrete-time version is in appendix B.

**Phillips curve.** The Phillips curve arises from households’ optimal pricing decisions:

$$\dot{\pi}(t) = \delta \pi(t) - \frac{\epsilon \kappa}{\gamma a} [y(t) - y^n],$$

where

$$y^n = \frac{\epsilon - 1}{\epsilon} \cdot \frac{a}{\kappa}.$$
The Phillips curve is not modified by wealth in the utility function. The steady-state Phillips curve, obtained by setting $\dot{\pi} = 0$ in (7), describes inflation as a linearly increasing function of output:

$$\pi = \frac{\varepsilon \kappa}{\delta y a} (y - y^n).$$

We see that $y^n$ is the natural level of output: the level at which producers keep their prices constant.

**Euler equation.** The Euler equation arises from households' optimal saving decisions:

$$\frac{\dot{y}(t)}{y(t)} = r(t) - r^n + u'(0) [y(t) - y^n],$$

where $r(t)$ is the real interest rate—determined by monetary policy—and

$$r^n = \delta - u'(0) y^n.$$  

As usual the financial return on saving, given by the real interest rate, is a key determinant of optimal saving and thus of the growth rate of consumption, $\dot{y}(t)/y(t)$. But unlike in the standard model, people also enjoy holding wealth, so a new term appears in the equation to capture the hedonic return on saving: the marginal rate of substitution between wealth and consumption, given by $u'(0)y(t)$. In the marginal rate of substitution, the marginal utility of wealth is $u'(0)$ because in equilibrium all households hold the same wealth so relative wealth is zero. Also in the marginal rate of substitution, the marginal utility of consumption is $1/y(t)$ because consumption utility is log.

The steady-state Euler equation is obtained by setting $\dot{y} = 0$ in (10):

$$u'(0)(y - y^n) = r^n - r.$$  

When $u'(0) > 0$, the equation describes output as a linearly decreasing function of the real interest rate—as in the old-fashioned, Keynesian IS curve. We also see that $r^n$ is the natural rate of interest: the real rate at which households consume a quantity $y^n$.

**NK and WUNK models.** The wealth-in-the-utility assumption only adds one parameter to the equilibrium conditions: $u'(0)$. Below we compare the following two submodels:

**Definition 1.** The New Keynesian (NK) model has zero marginal utility of wealth: $u'(0) = 0$. The
wealth-in-the-utility New Keynesian (WUNK) model has sufficient marginal utility of wealth:

\[ u' (0) > \frac{\epsilon \kappa}{\delta \gamma a}. \]

The NK model is the standard model; the WUNK model is the extension proposed in this paper. When prices are completely fixed \((\gamma \to \infty)\), condition (13) simply becomes \(u'(0) > 0\); when prices are perfectly flexible \((\gamma = 0)\), condition (13) becomes \(u'(0) > \infty\). Hence, at the fixed-price limit, the WUNK model only requires an infinitesimal marginal utility of wealth; at the flexible-price limit, the WUNK model is not well-defined. In the WUNK model we also impose \(\delta > \sqrt{\epsilon - 1}/\gamma\), in order to accommodate positive natural rates of interest while respecting (13).

2.4. Heuristic derivation

Our model incorporates several nontraditional elements (wealth in the utility function, and to a lesser extent continuous time and Rotemberg pricing). Thus, before delving into the analysis, we provide a heuristic derivation of the Euler equation and Phillips curve (following Blanchard and Fischer 1989, pp. 40–42).

Euler equation. The Euler equation says that households save in an optimal fashion: they cannot improve their situation by shifting consumption a little bit across time.

Consider a household delaying consumption of one unit of output from time \(t\) to time \(t + dt\). The unit of output, invested at a real rate \(r(t)\) during the interval, becomes \(1 + r(t)dt\) at time \(t + dt\). Given the utility function (5), the marginal utility from consumption at time \(t\) is \(e^{-\delta t}/y(t)\). Hence, the foregone consumption utility at time \(t\) is \(e^{-\delta t}/y(t)\); and the extra consumption utility at time \(t + dt\) is \(e^{-\delta (t+dt)} [1 + r(t)dt]/y(t + dt)\).

In addition to financial returns, the one unit of output saved between \(t\) and \(t + dt\) provides hedonic returns, since people enjoy holding wealth. The marginal utility from real wealth at time \(t\) is \(e^{-\delta t} u'(0)\); hence the utility gained from holding extra wealth for a duration \(dt\) is \(e^{-\delta t} u'(0) dt\).

At the optimum, the consumption reallocation does not affect utility:

\[
0 = -\frac{e^{-\delta t}}{y(t)} + [1 + r(t)dt] \frac{e^{-\delta (t+dt)}}{y(t + dt)} + e^{-\delta t} u'(0) dt.
\]

After dividing by \(e^{-\delta t}/y(t)\), we obtain

\[
1 = [1 + r(t)dt] e^{-\delta dt} \frac{y(t)}{y(t + dt)} + u'(0) y(t) dt.
\]
Up to second-order terms, \( e^{-\delta dt} = 1 - \delta dt \),

\[
\frac{y(t + dt)}{y(t)} = 1 + \frac{\dot{y}(t)}{y(t)} dt,
\]

and \( \frac{1}{1 + x dt} = 1 - x dt \) for any \( x \). Hence, up to second-order terms, we have

\[
1 = \left[ 1 + r(t) dt \right] (1 - \delta dt) \left[ 1 - \frac{\dot{y}(t)}{y(t)} dt \right] + u'(0)y(t) dt
\]

\[
= 1 - \delta dt + r(t) dt - \frac{\dot{y}(t)}{y(t)} dt + u'(0)y(t) dt,
\]

which simplifies to

\[
\frac{\dot{y}(t)}{y(t)} = r(t) - \delta + u'(0)y(t).
\]

(14)

Once we introduce \( r^n = \delta - u'(0)y^n \), we obtain (10).

Compared with the standard Euler equation, (14) has an extra term: \( u'(0)y(t) \). In the standard equation, consumption is governed by the cost of delaying consumption, given by the time discount rate \( \delta \), and the return on saving, given by the real interest rate \( r(t) \). With wealth in the utility function, the financial return on saving is supplemented by an hedonic return on saving, measured by the marginal rate of substitution between real wealth and consumption, \( u'(0)y(t) \). Thus the total return on saving is \( r(t) + u'(0)y(t) \) instead of \( r(t) \), explaining the extra term in the equation. Because consumption depends not only on interest rates but also on the marginal rate of substitution between wealth and consumption, future interest rates have less impact on today’s consumption. In fact, appendix B shows that the discrete-time Euler equation is discounted exactly as in McKay, Nakamura, and Steinsson (2017), and that discounting is stronger with higher marginal utility of wealth.

The derivation also explains why in steady state consumption is a decreasing function of the real interest rate. When the real rate is higher, people have a financial incentive to save more and postpone consumption. People keep consumption constant only if the hedonic return on saving falls so as to compensate the increase in the financial return. The hedonic return is given by the marginal rate of substitution \( u'(0)y \). In steady state consumption \( y \) must therefore decline when the real rate increases.

**Phillips curve.** The Phillips curve says that households price optimally, so they cannot improve their situation by shifting inflation a little bit across time.

Consider a household delaying one percentage point of inflation from time \( t \) to time \( t + dt \).
Given the utility function (5), the marginal disutility from inflation at time $t$ is $e^{-\delta t} y \pi(t)$. Hence, the foregone inflation disutility at time $t$ is $e^{-\delta t} y \pi(t) \times 1\%$; and the extra inflation disutility at time $t + dt$ is $e^{-\delta(t+dt)} y \pi(t + dt) \times 1\%$.

The one percentage point of inflation that is delayed reduces the price level between time $t$ and time $t + dt$ by $dp(t) = -1\% \times p(t)$, which then affects sales and hours worked. Since the price elasticity of demand is $-\epsilon$, sales increase by $dy(t) = -\epsilon \times -1\% \times y(t) = \epsilon y(t) \times 1\%$ during the same period. As a result, hours increase by $dh(t) = dy(t)/a = \epsilon[y(t)/a] \times 1\%$, raising the disutility of labor by $e^{-\delta t} \kappa \epsilon[y(t)/a] \times 1\%$ between $t$ and $t + dt$. The change in price and sales raises the revenue by $d(p(t)y(t)) = p(t)dy(t) + y(t)dp(t) = (\epsilon - 1)y(t)p(t) \times 1\%$. Since in equilibrium all prices are the same, the increase in revenue yields an increase in consumption $dy(t) = (\epsilon - 1)y(t) \times 1\%$, which raises consumption utility by $e^{-\delta t} dy(t)/y(t) = e^{-\delta t} (\epsilon - 1) \times 1\%$ between time $t$ and time $t + dt$.

At the optimum, shifting inflation across time does not affect utility:

$$0 = e^{-\delta t} y \pi(t) \times 1\% - e^{-\delta(t+dt)} y \pi(t + dt) \times 1\% - e^{-\delta t} \kappa \epsilon \frac{y(t)}{a} \times 1\% \times dt + e^{-\delta t} (\epsilon - 1) \times 1\% \times dt.$$ 

Up to second-order terms, $e^{-\delta dt} = 1 - \delta dt$ and

$$\pi(t + dt) = \pi(t) + \dot{\pi}(t) dt.$$

Therefore, after dividing by $e^{-\delta t} \times 1\%$, we find that up to second-order terms,

$$0 = y \pi(t) - (1 - \delta dt) y \left[ \pi(t) + \dot{\pi}(t) dt \right] - \kappa \epsilon \frac{y(t)}{a} dt + (\epsilon - 1) dt$$

$$= \delta y \pi(t) dt - y \dot{\pi}(t) dt - \kappa \epsilon \frac{y(t)}{a} dt + (\epsilon - 1) dt.$$ 

Dividing by $y dt$, we obtain up to second-order terms:

$$\dot{\pi}(t) = \delta \pi(t) - \frac{\epsilon \kappa}{y a} \left[ y(t) - \frac{\epsilon - 1}{\epsilon} \cdot a \right].$$

Once we introduce $y^n = [(\epsilon - 1)/\epsilon]a/\kappa$, we obtain (7).

The Phillips curve implies that in the absence of price-adjustment cost ($\gamma = 0$), households would like to produce at the natural level of output, $y^n$. This result comes from the monopolistic nature of competition. Without price-adjustment costs, it is optimal to charge a relative price that is a markup $\epsilon/(\epsilon - 1)$ over the real marginal cost—which is the marginal rate of substitution between labor and consumption divided by the marginal product of labor. In equilibrium, all relative
prices are 1, the marginal rate of substitution between labor and consumption is $\kappa/(1/y) = \kappa y$, and the marginal product of labor is $a$. Hence at the optimum $1 = [(\epsilon/(\epsilon - 1)]\kappa y/a$, which implies $y = [(\epsilon - 1)/\epsilon] a/\kappa = y^n$.

Last, the derivation elucidates why in steady state, inflation is positive whenever output is above its natural level. When inflation is positive, reducing inflation lowers the disutility from price adjustment. Since pricing is optimal, there must also be a cost to reducing inflation and hence increasing production. Therefore, production must already be excessive: output must be above its natural level.

2.5. Monetary policy and natural rate of interest

The central bank aims to maintain the economy at the natural steady state, where inflation is at zero and output at its natural level.

In normal times, the natural rate of interest is positive, and the central bank is able to maintain the economy at the natural steady state using a simple interest-rate rule:

$$i(\pi(t)) = r^n + \phi \pi(t).$$

The parameter $\phi \geq 0$ governs the response of monetary policy to inflation: monetary policy is active when $\phi > 1$ and passive when $\phi < 1$. The rule implies implies a real interest rate

$$r(\pi(t)) = r^n + (\phi - 1) \pi(t).$$

When the natural rate of interest is negative, however, the natural steady state cannot be achieved—because this would require a negative nominal interest rate, which would violate the ZLB. In that case, the central bank moves to the ZLB:

$$i(t) = 0 \quad \text{and} \quad r(t) = -\pi(t).$$

Why could cause the natural rate of interest to be negative? In the NK model, it is actually problematic to have a negative natural rate of interest while maintaining a positive time discount rate, since $r^n = \delta$. We address this issue as in Woodford (2011, p. 16): we assume that financial intermediation creates a spread $\sigma$ between the central bank’s interest rate, which is $i(t)$, and the interest rate used by households for saving decisions, which becomes $i(t) + \sigma$. Then the natural rate of interest—the real interest rate set by the central bank at which households consume $y^n$—becomes $r^n = \delta - \sigma$. If the spread is large ($\sigma > \delta$), the natural rate of interest becomes negative while the time discount rate remains positive. In the WUNK model, things are more
straightforward. The natural rate of interest is negative when that the marginal utility of wealth is high enough: \( u'(0) > \frac{\delta}{y^n} \).

### 2.6. Dynamics

We now describe dynamics in the NK and WUNK models. We first describe normal times: the natural rate of interest is positive, and monetary policy follows \( i(\pi) = r^n + \phi \pi \).

**Proposition 1.** Consider the NK and WUNK models in normal times. They admit a unique steady state, where output is at its natural level \( y = y^n \), inflation is zero \( \pi = 0 \), and the ZLB is not binding \( i = r^n > 0 \). Around this natural steady state, dynamics are governed by the linear dynamical system

\[
\begin{bmatrix}
\dot{y}(t) \\
\dot{\pi}(t)
\end{bmatrix}
= \begin{bmatrix}
u'(0)y^n & (\phi - 1)y^n \\
-\epsilon \kappa / (\gamma a) & \delta
\end{bmatrix}
\begin{bmatrix}
y(t) - y^n \\
\pi(t)
\end{bmatrix}.
\]

In the NK model, the dynamical system is a source when monetary policy is active \( \phi > 1 \) and a saddle when monetary policy is passive \( \phi < 1 \). In the WUNK model, the dynamical system is a source whether monetary policy is active or passive.

We next turn to dynamics at the ZLB: the natural rate of interest is negative, and monetary policy is \( i = 0 \).

**Proposition 2.** Consider the NK and WUNK models at the ZLB. They admit a unique steady state, where output and inflation are given by

\[
\begin{align}
y^z &= y^n + \frac{r^n}{u'(0) - \epsilon \kappa / (\delta \gamma a)} \\
\pi^z &= \frac{r^n}{u'(0) \delta \gamma a / (\epsilon \kappa) - 1}.
\end{align}
\]

Around the steady state, dynamics are governed by the linear dynamical system

\[
\begin{bmatrix}
\dot{y}(t) \\
\dot{\pi}(t)
\end{bmatrix}
= \begin{bmatrix}
u'(0)y^z & -y^z \\
-\epsilon \kappa / (\gamma a) & \delta
\end{bmatrix}
\begin{bmatrix}
y(t) - y^z \\
\pi(t) - \pi^z
\end{bmatrix}.
\]

In the NK model, the steady state has positive inflation \( \pi^z = -r^n > 0 \) and above-natural output \( y^z > y^n \), and the dynamical system is a saddle. In the WUNK model, the steady state has deflation \( \pi^z < 0 \) and below-natural output \( y^z < y^n \), and the dynamical system is a source.
Proof. We offer a graphical proof based on the diagrams in Figure 1. (To reduce the number of diagrams, we only discuss active monetary policy in normal times; passive monetary policy can be analyzed similarly.) This graphical approach nicely illustrates the mechanisms behind the results. An alternative, algebraic proof—closer to the proofs in the literature—is presented in Appendix C.

Steady states. A steady state satisfies both the steady-state Phillips curve, given by (9), and the steady-state Euler equation, given by (12). These steady-state equations are represented by the lines labeled “Phillips” and “Euler” in Figure 1. The Phillips line is the same in the NK and WUNK models, and in normal times and at the ZLB: it is upward-sloping and goes through the point \([y = y^n, \pi = 0]\). The Euler line, on the other hand, is different in each case.

In the NK model the Euler line is horizontal because when \(u'(0) = 0\), (12) imposes that \(r(\pi) = r^n\), which determines inflation independently of output (panels A and C). In the WUNK model, the Euler line is not horizontal because when \(u'(0) > 0\), (12) makes output a decreasing function of the real rate \(r(\pi)\). When monetary policy is active \((\phi > 1)\), \(r(\pi) = r^n + (\phi - 1)\pi\) is increasing in \(\pi\), so the Euler line slopes downward (panel B); at the ZLB, \(r(\pi) = -\pi\) is decreasing in \(\pi\), so the Euler line slopes upward (panel D).

The Euler line also changes between normal times and the ZLB because the natural rate of interest and monetary policy change. In the NK model the Euler line shifts up from \(\pi = 0\) in normal times to \(\pi = -r^n > 0\) at the ZLB (panels A and C). In the WUNK model, in normal times with active monetary policy, the Euler line is

\[
\pi = -\frac{u'(0)}{\phi - 1}(y - y^n),
\]

so it is downward sloping and goes through the point \([y = y^n, \pi = 0]\) (panel B). At the ZLB the Euler line is

\[
\pi = -r^n + u'(0)(y - y^n),
\]

so it is upward sloping and goes through the point \([y = y^n + r^n/u'(0), \pi = 0]\) (panel D). Since \(r^n \leq 0\) at the ZLB, the ZLB Euler line is always inward of the point \([y = y^n, \pi = 0]\), explaining why the central bank is unable to achieve the natural steady state. Further, since the slope of the Phillips line is \(\kappa/(\delta a)\) and the slope of the ZLB Euler line is \(u'(0)\), the WUNK condition (13) ensures that the ZLB Euler line is steeper than the Phillips line.

In normal times, in both models, the Phillips and Euler lines intersect at the point \([y = y^n, \pi = 0]\); therefore, the models admit a unique steady state with zero inflation and natural
output (panels A and B). At the ZLB, in both models, the Phillips and Euler lines have a unique intersection, so the steady state exists and is unique. In the NK model, the steady state has positive inflation and above-natural output (panel C); in the WUNK model, the steady state has negative inflation and below-natural output (panel D).

**Dynamics.** The global dynamics of the NK and WUNK models are governed by the dynamical system generated by the Phillips curve (7) and Euler equation (10), but around the steady states the dynamics can be obtained from linearized versions of this system. The linearization is particularly straightforward because (7) is already linear, and (10) can be immediately linearized as \( \dot{y} = \ddot{y} [r(\pi) - r^n + u'(0)(y - y^n)] \), where \( \ddot{y} \) stands for steady-state output. Combining these two linear differential equations and the appropriate expressions for \( r(\pi) \) and \( \ddot{y} \), we obtain the linear dynamical systems (15) and (18).

The dynamics produced by these linear systems are described by their phase diagrams. The first step in constructing these diagrams is to plot the \( y \)-nullcline (the locus \( \ddot{y} = 0 \)) and the \( \pi \)-nullcline (the locus \( \dddot{\pi} = 0 \)) given by (15) and (18). As the \( \pi \)-nullcline reduces to the steady-state Phillips curve (9), and the \( y \)-nullcline to the steady-state Euler equation (12), these nullclines are given by the Phillips and Euler lines in figure 1.

Second, we plot in figure 1 the arrows giving the directions of the trajectories in the phase diagrams. We first determine the sign of \( \ddot{\pi} \) in the linear systems. The differential equation giving \( \ddot{\pi} \) is just the Phillips curve (7); it shows that any point above the Phillips line (where \( \ddot{\pi} = 0 \)) has \( \ddot{\pi} > 0 \), and any point below the line has \( \ddot{\pi} < 0 \). So in all the panels of figure 1 we indicate that inflation is rising above the Phillips line and falling below it.

Next we examine the sign of \( \dot{y} \) in the linear systems (15) and (18). In the NK model, in normal times with active monetary policy (panel A), the differential equation giving \( \dot{y} \) is

\[
\frac{\dot{y}}{y^n} = (\phi - 1)\pi
\]

with \( \phi > 1 \). Hence any point above the Euler line (where \( \pi = 0 \)) has \( \dot{y} > 0 \), and any point below the line has \( \dot{y} < 0 \). Accordingly, in the four quadrants delimited by the Phillips and Euler lines, the trajectories move away from the steady state: the dynamical system a source.

In the NK model at the ZLB (panel C), the differential equation giving \( \dot{y} \) is

\[
\frac{\dot{y}(t)}{y^n} = -\pi - r^n.
\]

\(^3\text{In the WUNK model we also need to check that the intersection has positive output; appendix C shows that is always the case.}\)
Figure 1. Phase diagrams in normal times and at the ZLB in the NK and WUNK models

$\pi$ is inflation; $y$ is output; $y^\pi$ is the natural level of output; the Euler line is the $y$-nullcline (the locus $\dot{y} = 0$); and the Phillips line is the $\pi$-nullcline (the locus $\dot{\pi} = 0$). The NK model is the standard New Keynesian model. The WUNK model is the same as the NK model, except that wealth enters the utility function, and the marginal utility of wealth is sufficiently large to satisfy (13). In normal times, the natural rate of interest ($r^\pi$) is positive, and the nominal interest rate is given by $i = r^\pi + \phi\pi$; when monetary policy is active, $\phi > 1$. At the ZLB, the natural rate of interest is negative, and the nominal interest rate is zero. The phase diagrams in panels A and B represent the dynamical system (15), which is obtained by linearizing the Phillips curve and Euler equation around the natural steady state. The phase diagrams in panels C and D represent the dynamical system (18), which is obtained by linearizing the Phillips curve and Euler equation around the ZLB steady state. The phase diagrams are constructed using standard methodology for planar linear systems, as explained in the text. Panel A shows that in the NK model, in normal times, the dynamical system is a source when monetary policy is active. Panel C shows that in the NK model the dynamical system is a saddle at the ZLB. Panels B and D show that in the WUNK model the dynamical system is a source both in normal times and at the ZLB.
\( \pi \) is inflation; \( y \) is output; \( y^n \) is the natural level of output; the Euler line is the \( y \)-nullcline; and the Phillips line is the \( \pi \)-nullcline. The NK model is the standard New Keynesian model. The WUNK model is the same as the NK model, except that wealth enters the utility function, and the marginal utility of wealth is sufficiently large to satisfy (13). In normal times, the natural rate of interest \( (r^n) \) is positive, and the nominal interest rate is given by \( i = r^n + \phi \pi \); when monetary policy is active, \( \phi > 1 \). At the ZLB, the natural rate of interest is negative, and the nominal interest rate is zero. The trajectories in panels A and B solve the dynamical system (15), which is obtained by linearizing the Phillips curve and Euler equation around the natural steady state. The trajectories in panels C and D solve the dynamical system (18), which is obtained by linearizing the Phillips curve and Euler equation around the ZLB steady state. All the trajectories are constructed using standard methodology for planar linear systems, as explained in the text. Panel A shows that in the NK model, in normal times, the dynamical system is a source when monetary policy is active. Panel C shows that in the NK model the dynamical system is a saddle at the ZLB. Panels B and D show that in the WUNK model the dynamical system is a source both in normal times and at the ZLB. In panels A and B, we have plotted a nodal source, but the system could also be a spiral source, depending on the value of \( \phi \); in panel D the system is always a nodal source.

**Figure 2. Trajectories in normal times and at the ZLB in the NK and WUNK models**
Hence any point above the Euler line (where $\pi = -r^n$) has $\dot{y} < 0$, and any point below the line has $\dot{y} > 0$. We conclude that the dynamical system is a saddle, because in the southwest and northeast quadrants the trajectories move toward the steady state, while in the southeast and northwest quadrants the trajectories move away from the steady state.

In the WUNK model, in normal times, and with active monetary policy (panel B), the differential equation giving $\dot{y}$ is

$$\frac{\dot{y}}{y^n} = (\phi - 1)\pi + u'(0)(y - y^n)$$

with $\phi > 1$. Accordingly, any point above the Euler line (where $\dot{y} = 0$) has $\dot{y} > 0$, and any point below the line has $\dot{y} < 0$. As in all four quadrants the trajectories move away from the steady state, the dynamical system is a source.

Last, in the WUNK model at the ZLB (panel D), the differential equation giving $\dot{y}$ is

$$\frac{\dot{y}}{y^z} = -\pi - r^n + u'(0)(y - y^n).$$

Any point above the Euler line (where $\dot{y} = 0$) has $\dot{y} < 0$, and any point below the line has $\dot{y} > 0$. Thus in all four quadrants the trajectories move away from the steady state: the dynamical system remains a source.

Overall, the key difference between the NK and WUNK models is that at the ZLB, the dynamical system representing the equilibrium remains a source in the WUNK model, whereas it becomes a saddle in the NK model. This difference will explain why the WUNK model does not suffer from the anomalies plaguing the NK model at the ZLB.

**Trajectories.** To complement the phase diagrams of figure 1, we plot the associated trajectories in figure 2. The plots show how the trajectories escape the steady state when the system is a source, and how they converge to the steady state along the saddle path when the system is a saddle. When the system is a source, the trajectories are organized around two unstable lines—trajectories that move away from the steady state in a straight line. At $t \to -\infty$, all the trajectories leave the steady state and are tangent to the same unstable line. At $t \to +\infty$, all the trajectories move to infinity and are parallel to the other unstable line. When the system is a saddle, there is one stable line, which goes to the steady state in a straight line, and one unstable line, which moves away from the steady state in a straight line. All trajectories are parallel to the stable line when $t \to -\infty$ and are parallel to the unstable line when $t \to +\infty$. Appendix C explains how the stable and unstable lines are linked to the eigenvectors and eigenvalues of the matrices in (15) and (18).

The phase diagrams illustrate the origin of the WUNK condition (13). The dynamical system
for the WUNK model remains a source at the ZLB as long as the Euler line is steeper than the Phillips line (panels D of figures 1 and 2). The Euler line’s slope at the ZLB is the marginal utility of wealth, so that marginal utility is required to be above a certain level—which is given by (13).

Propositions 1 and 2 have implications for equilibrium determinacy. When the system is a source, the equilibrium is determinate: the only equilibrium trajectory in the vicinity of the steady state is to jump to the steady state and stay there; if the economy jumped somewhere else, output or inflation would diverge following a trajectory similar to those plotted in panels A, B, and D of figure 2. When the system is a saddle, in contrast, the equilibrium is indeterminate: any trajectory jumping somewhere on the saddle path and converging to the steady state is an equilibrium, as illustrated in panel C of figure 2. Hence, in the NK model, the equilibrium is indeterminate in normal times when monetary policy is passive, and at the ZLB. In the WUNK model, in contrast, the equilibrium is always determinate, even when monetary policy is passive in normal times and at the ZLB.

Accordingly, the Taylor principle holds in the NK model: the equilibrium is determinate only when monetary policy is active. This is not the case in the WUNK model: the equilibrium is determinate whether monetary policy is active or passive. Thus, monetary policy plays quite a different role in the two models. In the NK model, the central bank must adhere to an active monetary policy to avoid indeterminacy. In the WUNK model, the central bank never needs to worry about how strongly its interest-rate rule responds to inflation: indeterminacy is never a risk, so the central bank could simply use an interest-rate peg.

The NK results in the propositions are well-known (for example, Woodford 2001). The WUNK results are similar to the results obtained by Gabaix (2016). He finds that when bounded rationality is strong enough, the dynamical system generated by the Phillips curve and Euler equation is a source even at the ZLB. He also finds that when prices are more flexible, more bounded rationality is required to maintain the source property. The same is true here: higher marginal utility of wealth is required for condition (13) to hold when the price-adjustment cost γ is lower. The logic is illustrated in panel D of figure 1. The system remains a source at the ZLB only when the Euler line is steeper than the Phillips line. As prices are more flexible, the Phillips line becomes steeper, and the required steepness for the Euler line increases. As the slope of the Euler line is determined by bounded rationality in the Gabaix model and by marginal utility of wealth in our model, these need to be larger when prices are more flexible.
3. New Keynesian anomalies at the ZLB

In this section we illustrate the anomalies of the NK model at the ZLB: collapse of output and inflation when the ZLB is long lasting, and implausibly strong effects of forward guidance and government spending. We then show that these anomalies disappear in the WUNK model.

3.1. Output and inflation collapse

Scenario. We consider a temporary ZLB episode, as in Werning (2011). Between times 0 and $T > 0$, a negative aggregate-demand shock brings the natural rate of interest below zero. Such a shock affects the Euler equation but not the Phillips curve: in the NK model it would be an increase in the financial-intermediation spread, reflecting a disruption in the financial sector; in the WUNK model it would be an increase in the marginal utility of wealth, reflecting a lower appetite for consumption and a higher appetite for saving. In response to the shock, the central bank maintains the nominal interest rate at zero. Then after time $T$, the shock subsides, the natural rate becomes positive again, and the central bank returns to normal monetary policy.

NK model. We start with the NK model. We analyze the ZLB episode by going backward in time, using the phase diagrams in panels A and C of figure 3. After time $T$, monetary policy maintains the economy at the natural steady state. Since equilibrium trajectories must be continuous, the economy must be at the natural steady state at the end of the ZLB, when $t = T$.

We then move back to the ZLB episode, when $t < T$. At time 0, inflation and output jump to the unique position leading to $[y = y^n, \pi = 0]$ at time $T$. Hence, inflation and output initially jump down to $\pi(0) < 0$ and $y(0) < y^n$. After their initial collapse, inflation and output recover following the unique trajectory leading to $[y = y^n, \pi = 0]$. The ZLB therefore creates a slump, with below-natural output and deflation (panel A).

Critically, output and inflation are always on the same trajectory during the ZLB, irrespective of the ZLB duration $T$. A longer ZLB only forces output and inflation to start from a lower position on the trajectory at time 0. Thus, as the ZLB lasts longer, initial output and inflation become unboundedly low (panel C).

WUNK model. In the WUNK model output and inflation never collapse during the ZLB, as shown by the phase diagrams in panels B and D of figure 3. Initially inflation and output jump down toward the ZLB steady state, but not all the way: $\pi^z < \pi(0) < 0$ and $y^z < y(0) < y^n$. Then they recover, following the unique trajectory going through $[y = y^n, \pi = 0]$. Consequently the ZLB episode creates a slump (panel B), which is deeper when the ZLB lasts longer (panel D).
But unlike in the NK model, output and inflation are bounded below: irrespective of the length of the ZLB, they always fall less than if the ZLB were permanent. Moreover, if the natural rate of interest is negative but close to 0, such that $\pi^z$ is close to 0 and $y^z$ to $y^n$, output and inflation barely deviate from the natural steady state during the ZLB, even if the ZLB lasts a very long time.

**Summary and discussion.** The following proposition records these results.\(^4\)

**Proposition 3.** Consider a ZLB episode between times 0 and $T$ in the NK and WUNK models. The economy enters a slump: $y(t) < y^n$ and $\pi(t) < 0$ for all $t \in (0, T)$. In the NK model, the slump becomes infinitely severe as the ZLB becomes infinitely long: $\lim_{T \to \infty} y(0) = \lim_{T \to \infty} \pi(0) = -\infty$.

In the WUNK model, in contrast, the slump is bounded below by the ZLB steady state: $y(t) > y^z$ and $\pi(t) > \pi^z$ for all $t \in (0, T)$. In fact, the slump approaches the ZLB steady state as the ZLB becomes infinitely long: $\lim_{T \to \infty} y(0) = y^z$ and $\lim_{T \to \infty} \pi(0) = \pi^z$.

In the NK model, output and inflation collapse when the ZLB is long-lasting—which is well-known (Eggertsson and Woodford 2004, fig. 1; Eggertsson 2011, fig. 1; Werning 2011, proposition 1). This collapse is difficult to reconcile with real-world observations. The ZLB episode that started in 1995 in Japan has been lasting for more than twenty years without sustained deflation. The ZLB episode that started in 2009 in the euro area has been lasting for more than 10 years; it did not yield sustained deflation either. The same is true of the ZLB episode that occurred in the United States between 2008 and 2015.

In the WUNK model, in contrast, the ZLB slump is bounded below by the ZLB steady state. So inflation and output never collapse at the ZLB, even if the ZLB lasts a very long time. Instead, as the duration of the ZLB increases, the economy converges to the ZLB steady state. And the ZLB steady state may not be far from the natural steady state: if the natural rate of interest is only slightly negative at the ZLB, steady-state inflation is only slightly below zero and steady-state output only slightly below its natural level (see (16) and (17)).

Gabaix (2016) obtains the results closest to those in proposition 3. In his model output and inflation also converge to the ZLB steady state as the ZLB becomes arbitrarily long.

### 3.2. Forward guidance

We turn to the effects of forward guidance at the ZLB.

---

\(^4\)The result that output becomes infinitely negative when the ZLB becomes infinitely long should not be interpreted literally. It is obtained from the dynamical system (18), which is a local approximation of the dynamical system given by (7) and (10). The interpretation is that output falls much below its natural level—so much so that the local approximation stops being valid. The global dynamical system guarantees that output remains positive.
A. NK model: short ZLB

B. WUNK model: short ZLB

C. NK model: long ZLB

D. WUNK model: long ZLB

**Figure 3. ZLB episodes of various durations in the NK and WUNK models**

The ZLB is binding between times 0 and \( T \); then, at time \( T \), the central bank brings the economy to the natural steady state, where inflation is zero and output is at its natural level. The phase diagrams describe dynamics at the ZLB; they come from panels C and D in figure 1. The equilibrium trajectories are the unique trajectories reaching the natural steady state at time \( T \). In the NK and WUNK models, the economy slumps during the ZLB: inflation is negative and output is below its natural level. In the NK model, the initial slump becomes unboundedly large as the ZLB becomes longer. In the WUNK model, there is no such output and inflation collapse: the initial slump is bounded below by the ZLB steady state.
**Scenario.** We consider a three-stage scenario, as in Cochrane (2017). Between times 0 and $T$, there is a ZLB episode, exactly as in the previous section. To alleviate the ZLB, the central bank makes a forward-guidance promise at time 0: that it will maintain the nominal interest rate at zero for a duration $\Delta$ once the ZLB is over. At time $T$, the adverse shock that brought the economy to the ZLB recedes, and the natural rate of interest returns above zero. Then, between times $T$ and $T + \Delta$, the central bank abides by its forward-guidance promise and keeps the nominal interest rate at zero. Finally, after time $T + \Delta$, monetary policy returns to normal.

**NK model.** Forward guidance in the NK model is analyzed in figure 4, by going backward in time. After time $T + \Delta$, monetary policy maintains the economy at the natural steady state, and so the economy must be at the natural steady state at the end of forward guidance, when $t = T + \Delta$. Between times $T$ and $T + \Delta$, the economy is in forward guidance, as depicted in panel A. The phase diagram is the same as in panel C of figure 1, except that $r^n > 0$ instead of $r^n < 0$: the Euler line, given by $\pi = -r^n$, is lower; but the system remains a saddle. Following the logic of figure 3, we find that at time $T$, inflation must be positive and output above its natural level. They subsequently decrease over time, following the unique trajectory leading to the natural steady state at time $T + \Delta$. Accordingly, the economy booms during forward guidance: inflation is always positive and output above its natural level. Furthermore, as the duration of forward guidance increases, inflation and output at the beginning of forward guidance rise.

We look next at the ZLB episode, between times 0 and $T$. This episode is depicted in panels B, C, and D; the three panels differ by the duration of forward guidance after the ZLB episode. Since equilibrium trajectories are continuous, the economy must be at the same point at the end of the ZLB and at the beginning of forward guidance. Because of the boom engineered during forward guidance, then, the situation is improved at the ZLB. Instead of reaching the natural steady state at time $T$, the economy reaches a point with positive inflation and above-natural output, so at any time before $T$, inflation and output tend to be higher than without forward guidance.

Forward guidance can have tremendously strong effects. For small durations of forward guidance, the position at the beginning of forward guidance is below the unstable line of the dynamical system representing the ZLB equilibrium. It is therefore connected to trajectories coming from the southwest, slumpy quadrant of the phase diagram (panel B). As the ZLB lasts longer, initial output and inflation collapse. When the duration of forward guidance is such that the position at the beginning of forward guidance is exactly on the unstable line, then the position at the beginning of the ZLB must be on the unstable line as well (panel C). As the ZLB lasts longer, the initial position inches closer to the ZLB steady state. For even longer forward guidance, the position at the beginning of forward guidance is above the unstable line, so it is connected to
The ZLB episode lasts between times 0 and $T$. It is followed by forward guidance between times $T$ and $T + \Delta$: the natural rate of interest becomes positive but the central bank maintains the nominal interest rate at zero. Then, at time $T + \Delta$, the central bank brings the economy to the natural steady state. The phase diagram in panel A describes dynamics during forward guidance; it is similar to the diagram in panel C of figure 1 but with $r^n > 0$. The phase diagrams in panels B, C, and D describe dynamics at the ZLB; they come from panels C of figures 1 and 2. The equilibrium trajectory during the forward-guidance episode is the unique trajectory reaching the natural steady state at time $T + \Delta$. The equilibrium trajectory at the ZLB is the unique trajectory reaching the point determined by forward guidance at time $T$. The NK model suffers from a major anomaly: when forward guidance lasts sufficiently to bring $[y(T), \pi(T)]$ on the right-hand side of the unstable line, any ZLB episode—however long—will be a boom (panel B versus panel D).

**Figure 4.** NK model: ZLB followed by forward-guidance episodes of various durations
The ZLB episode lasts between times 0 and $T$. It is followed by forward guidance between times $T$ and $T + \Delta$: the natural rate of interest becomes positive but the central bank maintains the nominal interest rate at zero. Then, at time $T + \Delta$, the central bank brings the economy to the natural steady state. The phase diagram in panel A describes dynamics during forward guidance; it is similar to the diagram in panel D of figure 1 but with $r_n > 0$. The phase diagrams in panels B and C describe dynamics at the ZLB; they come from panel D of figure 1. Panel D is a generic version of panels B and C, describing any duration of ZLB and forward guidance. The equilibrium trajectory during the forward-guidance episode is the unique trajectory reaching the natural steady state at time $T + \Delta$. The equilibrium trajectory at the ZLB is the unique trajectory reaching the point determined by forward guidance at time $T$. The anomaly of the NK model disappears in the WUNK model, as a long-enough ZLB always leads to a slump (panels C and D versus panel B).
trajectories coming from the northeast, boomy quadrant of the phase diagram (panel D). From here, as the ZLB lasts longer, initial output and inflation become higher and higher. As a result, if the duration of forward guidance is long enough, a deep slump can be transformed into a roaring boom. Moreover, the effect of such policy are larger when the ZLB lasts longer, although the required duration of forward guidance is independent of the ZLB duration.

**WUNK model.** The power of forward guidance is subdued in the WUNK model, as illustrated in figure 5. After time $T + \Delta$, the economy is at the natural steady state. Between times $T$ and $T + \Delta$, forward guidance operates, as shown in panel A. The phase diagram is the same as in panel D of figure 1, except that $r^n > 0$ instead of $r^n < 0$: the Euler line (19) is shifted outward; yet, the dynamical system remains a source. Inflation must be positive and output must be above its natural level at time $T$; then they decrease over time, following the unique trajectory leading to the natural steady state at time $T + \Delta$. The economy booms during forward guidance; but unlike in the NK model, output and inflation are bounded above by the forward-guidance steady state.

Before forward guidance comes the ZLB episode, depicted in panels B and C. Thanks to the boom engineered by forward guidance, the situation is improved at the ZLB: inflation and output tend to be higher than without forward guidance. Yet, unlike in the NK model, output during the ZLB episode is always below its level at the beginning of forward guidance. So forward guidance cannot generate unbounded booms in the WUNK model. The ZLB cannot generate unbounded slumps either, since output and inflation are bounded below by the ZLB steady state. These properties are summarized in panel F. Finally, for any forward-guidance duration, as the ZLB lasts longer, the economy converges to the ZLB steady state at time $0$; thus, forward guidance can never prevent a slump when the ZLB lasts long enough.

**Summary and discussion.** Based on these dynamics, we isolate two anomalies in the NK model, which are resolved in the WUNK model (appendix C fleshes out the proof):

**Proposition 4.** Consider a ZLB episode during $(0, T)$ followed by forward guidance during $(T, T + \Delta)$.

- In the NK model, there exists a threshold $\Delta^*$, such that any forward guidance longer than $\Delta^*$ transforms a ZLB of any duration into a boom: let $\Delta > \Delta^*$; then for any $T$ and for all $t \in (0, T + \Delta)$, $y(t) > y^n$ and $\pi(t) > 0$. In addition, when the forward guidance is longer than $\Delta^*$, a long-enough forward guidance or a long-enough ZLB generates an arbitrarily large boom: for any $T$, $\lim_{\Delta \to \infty} y(0) = \lim_{\Delta \to \infty} \pi(0) = +\infty$; and for any $\Delta > \Delta^*$, $\lim_{T \to \infty} y(0) = \lim_{T \to \infty} \pi(0) = +\infty$. 


• In the WUNK model, in contrast, there exists a threshold $T^*$, such that any ZLB longer than $T^*$ generates a slump, irrespective of the duration of forward guidance: let $T > T^*$; then for any $\Delta$, $y(0) < y^n$ and $\pi(0) < 0$. Furthermore, the slump approaches the ZLB steady state as the ZLB becomes infinitely long: for any $\Delta$, $\lim_{T \to \infty} y(0) = y^z$ and $\lim_{T \to \infty} \pi(0) = \pi^z$. In addition, the economy is bounded above by the forward-guidance steady state $[y^f, \pi^f]$: for any $T$ and $\Delta$, and for all $t \in (0, T + \Delta)$, $y(t) < y^f$ and $\pi(t) < \pi^f$.

The anomalies identified in the proposition correspond to the instances of the forward-guidance puzzle described by Carlstrom, Fuerst, and Paustian (2015, fig. 1) and Cochrane (2017, fig. 6). These papers also find that a long-enough forward guidance transforms a ZLB slump into a boom whose amplitude increases with the ZLB duration.

In the WUNK model, such anomalous patterns vanish. In the New Keynesian models by Gabaix (2016), Diba and Loisel (2019), Acharya and Dogra (2019), and Bilbiie (2019), forward guidance also has much more subdued effects than in the standard model. Besides, New Keynesian models have been developed with the sole goal of solving the forward-guidance puzzle. Among these, ours belongs to the group that uses discounted Euler equations—in which future interest rates have less effect on today’s consumption than in the standard equation. For example, Del Negro, Giannoni, and Patterson (2015) generate discounting from overlapping generations; McKay, Nakamura, and Steinsson (2016) from heterogeneous agents facing borrowing constraints and cyclical income risk; and Angeletos and Lian (2018) from incomplete information. Closely related to our approach, Campbell et al. (2017) generate discounting by introducing government bonds in the utility function.

3.3. Government spending

Finally we consider the effects of government spending at the ZLB. For that we need to extend the model slightly. (The new derivations are relegated to appendix D.)

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5In the literature the forward-guidance puzzle takes several forms. Their common element is that future monetary policy has an implausibly strong effect on output and inflation today.

6Other approaches to solve the forward-guidance puzzle include modifying the Phillips curve (Carlstrom, Fuerst, and Paustian 2015), combining “reflective” expectations and temporary equilibrium (Garcia-Schmidt and Woodford 2019), combining bounded rationality and incomplete markets (Farhi and Werning 2019), or introducing an endogenous liquidity premium (Bredemeier, Kaufmann, and Schabert 2018).
**Assumptions.** We assume that the government purchases quantities \( g_k(t) \) of the goods \( k \in [0, 1] \). These quantities are aggregated into an index of public consumption

\[
g(t) \equiv \left[ \int_0^1 g_k(t)^{(\varepsilon-1)/\varepsilon} \, dk \right]^{\varepsilon/(\varepsilon-1)}.
\]

Public consumption \( g(t) \) enters separately into households’ utility functions. Government expenditure is financed with lump-sum taxation.

We also assume that the disutility of labor is convex—which implies a finite Frisch elasticity of labor supply. Household \( j \) incurs disutility

\[
\frac{\kappa^{1+\eta} h_j(t)^{1+\eta}}{1 + \eta}
\]

from working, where \( \eta > 0 \) is the inverse of the Frisch elasticity. The utility function is altered to ensure that government spending affects inflation and private consumption.

**Equilibrium.** In this extended model, once it is expressed in terms of private consumption, the Euler equation is the same as before:

\[
\frac{\dot{c}(t)}{c(t)} = r(\pi(t)) + u'(0)c(t) - \delta.
\]

On the other hand, the Phillips curve is modified:

\[
\dot{\pi}(t) = \delta \pi(t) + \frac{(\varepsilon-1)[c(t) + g(t)]}{yc(t)} \left\{ 1 - \frac{\varepsilon}{\varepsilon-1} \left( \frac{\kappa^{1+\eta}}{a} \right) [c(t) + g(t)]^\eta c(t) \right\}.
\]

The interpretation of the Phillips curve remains the same, except that the real marginal cost—the term after \( \varepsilon/(\varepsilon-1) \) in the curly brackets—takes a more complicated form, because the marginal disutility of labor takes a more complicated shape:

\[
\kappa^{1+\eta} h(t)^\eta = \kappa^{1+\eta} y(t)/a^\eta = \kappa^{1+\eta} [c(t) + g(t)]^\eta /a^\eta.
\]

Since the Phillips curve changes, we adjust the WUNK assumption. We replace (13) by

\[
u'(0) > (1 + \eta) \frac{\varepsilon \kappa}{\delta y a} \left( \frac{\varepsilon - 1}{\varepsilon} \right)^{\eta/(1+\eta)}.
\]

Naturally, for \( \eta = 0 \), this assumption reduces to (13).

Despite the changes, the model retains the same properties: propositions similar to propositions 1 and 2 hold (propositions A1 and A2 in the appendix). The main novelty is that additional
steady states may appear, because of the nonlinearity of the Phillips curve. In that case, we follow the literature and concentrate on the steady state closest to the natural steady state.

**Scenario.** We study a ZLB episode during which the government purchases goods to stimulate the economy, as in Cochrane (2017). Between times 0 and $T$, the ZLB binds: a negative aggregate demand shock makes the natural rate of interest negative, which prompts the central bank to keep the nominal interest rate at zero. To further alleviate the situation, the government provides an amount $g > 0$ of public consumption. After time $T$, the natural rate becomes positive again, so monetary policy returns to normal, and government spending stops: the economy returns to the natural steady state, with zero inflation and private consumption at $c^\text{n}$.

**Phase diagrams at the ZLB.** We analyze the effect of government spending using the phase diagrams in figure 6 (for the NK model) and figure 7 (for the WUNK model). The diagrams represent the linear dynamical system obtained by linearizing the Euler equation (21) and Phillips curve (22) around the natural steady state without government spending. The linearized Euler equation is

\[
\frac{\dot{c}(t)}{c^\text{n}} = u'(0) [c(t) - c^\text{n}] - \pi(t) - r^\text{n};
\]

and the linearized Phillips curve is

\[
\dot{\pi}(t) = \delta \pi(t) - \frac{\epsilon \kappa}{\gamma a} \left( \frac{\epsilon - 1}{\epsilon} \right)^{\eta/(1+\eta)} [(1 + \eta)(c(t) - c^\text{n}) + \eta g],
\]

where the natural level of consumption $c^\text{n}$ and natural rate of interest $r^\text{n}$ are given by

\[
c^\text{n} = \left( \frac{\epsilon - 1}{\epsilon} \right)^{1/(1+\eta)} \frac{a}{\kappa},
\]

\[
r^\text{n} = \delta - u'(0)c^\text{n}.
\]

We use these linear differential equations to determine the directions of the trajectories in the phase diagrams.

In steady state, the linearized Euler equation and Phillips curve become

\[
\pi = -\delta + u'(0)c
\]

\[
\pi = -\frac{\epsilon \kappa}{\delta \gamma a} \left( \frac{\epsilon - 1}{\epsilon} \right)^{\eta/(1+\eta)} [(1 + \eta)(c - c^\text{n}) + \eta g].
\]
We use these steady-state equations to draw the $c$-nullcline (Euler line) and $\pi$-nullcline (Phillips line) in the phase diagrams. As is typical in New Keynesian models, government spending operates by shifting up the Phillips line (see Werning 2011; Cochrane 2017).\textsuperscript{7}

**NK model.** We start with the NK model, illustrated in figure 6. We construct the paths of private consumption and inflation by going backward in time, given that at time $T$, monetary policy brings the economy to the natural steady state. During the ZLB, government spending helps, but through a different mechanism than forward guidance. Forward guidance better the situation at the end of the ZLB, which pulls up the economy during the entire ZLB. With government spending the end of the ZLB is unchanged: the economy reaches the natural steady state. But by shifting the Phillips line upward, and with it the field of trajectories, an increase in government spending connects the natural steady state to trajectories with higher consumption and inflation (panel A versus panel B).

Just like forward guidance, government spending can have very strong effects in the NK model. For low levels of spending (panel B), the natural steady state is below the unstable line of the dynamical system representing the ZLB equilibrium; it is therefore connected to trajectories coming from the southwest, slumpy quadrant of the phase diagram—just as without government spending (panel A). Then, if the ZLB lasts longer, initial consumption and inflation fall lower. When government spending is higher enough that the natural steady state is on the unstable line, the position at the beginning of the ZLB must also be on the unstable line (panel C). Then, as the ZLB lasts longer, the initial position moves closer to the ZLB steady state. Finally, when government spending is even higher, the natural steady state moves above the unstable line, so it is connected to trajectories coming from the northeast, boomy quadrant of the phase diagram (panel D). As a result, initial output and inflation are much higher than previously. And as the ZLB lasts longer, initial output and inflation become higher, without any upper bound.

**WUNK model.** The power of government spending at the ZLB is much weaker in the WUNK model, as shown in figure 7. After time $T$, the economy is at the natural steady state; prior to that comes the ZLB episode. Government spending improves the situation at the ZLB, as inflation and consumption tend to be higher than without spending. As the ZLB lasts longer, the position at the beginning of the ZLB converges to the ZLB steady state—unlike in the NK model, it does not diverge to infinity. So equilibrium trajectories are bounded, and government spending cannot generate unbounded booms.

\textsuperscript{7}When the disutility of labor is linear ($\eta = 0$), government spending has no first-order effect on the Phillips curve (since $\eta g = 0$), and government spending does not affect private consumption and inflation. This is why we assumed a convex disutility of labor ($\eta > 0$).
\( \pi \) is inflation; \( c \) is private consumption; \( c^* \) is the natural level of private consumption; the Euler line is the \( c \)-nullcline; and the Phillips line is the \( \pi \)-nullcline. The phase diagrams represent the linear dynamical system composed by (24) and (25) with \( u'(0) = 0 \); this system is obtained by linearizing the Phillips curve and Euler equation around the natural steady state without government spending. The phase diagrams have the same properties as those in panels C of figures 1 and 2, except that the Phillips line shifts upward as government spending increases. The ZLB episode lasts between times 0 and \( T \). During the ZLB, government spending is positive. Then, at time \( T \), government spending stops and the central bank brings the economy to the natural steady state. The equilibrium trajectory during the ZLB is the unique trajectory reaching the natural steady state at time \( T \). The NK model suffers from two anomalies: when government spending brings the unstable line below the natural steady state, any long-enough ZLB episode will see an arbitrarily large increase in output (panel B versus panel D); and then, any long-enough ZLB episode will experience an unboundedly large boom.

**Figure 6. NK model: ZLB episodes with various levels of government spending**

A. ZLB with no government spending

B. ZLB with low government spending

C. ZLB with medium government spending

D. ZLB with high government spending
A. ZLB with no government spending

B. ZLB with low government spending

C. ZLB with medium government spending

D. ZLB with high government spending

**Figure 7. WUNK model: ZLB episodes with various levels of government spending**

$\pi$ is inflation; $c$ is private consumption; $c^n$ is the natural level of private consumption; the Euler line is the $c$-nullcline; and the Phillips line is the $\pi$-nullcline. The phase diagrams represent the linear dynamical system composed by (24) and (25) with $u'(0)$ satisfying (23); this system is obtained by linearizing the Phillips curve and Euler equation around the natural steady state without government spending. The phase diagrams have the same properties as those in panels D of figures 1 and 2, except that the Phillips line shifts upward as government spending increases. The ZLB episode lasts between times 0 and $T$. During the ZLB, government spending is positive. Then, at time $T$, government spending stops and the central bank brings the economy to the natural steady state. The equilibrium trajectory during the ZLB is the unique trajectory reaching the natural steady state at time $T$. The anomalies of the NK model disappear in the WUNK model: output multipliers are finite when the ZLB becomes arbitrarily long-lasting; and irrespective of the duration of the ZLB, the equilibrium trajectories are always bounded.
Summary and discussion. Based on these dynamics, we isolate two anomalies in the NK model, which are resolved in the WUNK model (appendix E fleshes out the proof):

Proposition 5. Consider a ZLB episode accompanied by government spending $g > 0$ during $(0, T)$. Let $c(t; g)$ and $y(t; g)$ be private consumption and output at time $t$; let $s > 0$ be some incremental government spending; and let

$$m(g, s) = \frac{y(0; g + s/2) - y(0; g - s/2)}{s} = 1 + \frac{c(0; g + s/2) - c(0; g - s/2)}{s}$$

be the government-spending multiplier.

- In the NK model, there exists a government spending $g^*$ such that the government-spending multiplier becomes infinitely large when the ZLB is infinitely long-lasting: for any $s > 0$, $\lim_{T \to \infty} m(g^*, s) = +\infty$. In addition, when government spending is above $g^*$, a long-enough ZLB generates an arbitrarily large boom: for any $g > g^*$, $\lim_{T \to \infty} c(0; g) = +\infty$.

- In the WUNK model, in contrast, when the ZLB is infinitely long-lasting, the multiplier always has a finite limit: for any $g$ and $s$, when $T \to \infty$, $m(g, s)$ converges to

$$1 + \frac{\eta}{u'(0)\gamma a \cdot \left(\frac{1}{\epsilon - 1}\right)^{\eta/(1+\eta)} - (1 + \eta)}.$$

Moreover, for any ZLB duration, the economy remains bounded above: let $c^g$ be private consumption in the ZLB steady state with government spending $g$; then for any $T$ and for all $t \in (0, T)$, $c(t; g) < \max\{c^g, c^n\}$.

The anomaly that a finite amount of government spending may generate an infinitely large boom as the ZLB becomes arbitrarily long is reminiscent of the findings by Christiano, Eichenbaum, and Rebelo (two, fig. 2), Woodford (two, fig. 2), and Cochrane (two, fig. 5) that in the NK model, government spending is exceedingly powerful when the ZLB is long-lasting.

In the WUNK model, such anomaly vanishes. Diba and Loisel (two) and Acharya and Dogra (two) also obtain more realistic effects of government spending at the ZLB. Beside these papers, Bredemeier, Juessen, and Schabert (two) obtain moderate multipliers at the ZLB by introducing an endogenous liquidity premium in the New Keynesian model.

4. Other New Keynesian properties at the ZLB

Beside the anomalous properties described in the previous section, the New Keynesian model has several other intriguing properties at the ZLB: paradox of thrift, paradox of toil, paradox of
flexibility, and above-one government-spending multiplier. Here we show that the WUNK model shares these properties.

In the NK model a permanent ZLB is indeterminate, which has forced researchers, starting with Krugman (1998) and Eggertsson and Woodford (2003), to study these properties in the context of temporary ZLB. This is not the case in the WUNK model, so we can simply work with a permanent ZLB. We assume that the natural rate of interest is permanently negative, and the central bank maintains the nominal interest rate at zero. The only equilibrium is to be at the ZLB steady state, where the economy is in a slump: inflation is negative and output is below its natural level. The ZLB equilibrium is represented graphically in figure 8: it is the intersection of a Phillips line, describing the steady-state Phillips curve (9), and an Euler line, describing the steady-state Euler equation (19). When an unexpected and permanent shock occurs, the economy jumps to a new ZLB steady state. We use the graphs to describe the effects of such shocks.

4.1. Paradox of thrift

We first study an increase in the marginal utility of wealth ($u'(0)$). The steady-state Phillips curve is unaffected, but the steady-state Euler equation does change. Using (11), we rewrite the steady-state Euler equation (19):

\[ \pi = -\delta + u'(0)y. \]

Hence increasing the marginal utility of wealth steepens the Euler line, which moves the economy inward along the Phillips line: output and inflation decrease (figure 8, panel A). The following proposition summarizes the result:

**Proposition 6.** At the ZLB in the WUNK model, the paradox of thrift holds: an unexpected and permanent increase in the marginal utility of wealth reduces output and inflation but does not affect relative wealth.

The paradox of thrift was first discussed by Keynes, but it also appears in the New Keynesian model (Eggertsson 2010, p. 16; Eggertsson and Krugman 2012, p. 1486). When the marginal utility of wealth is higher, people want to increase their wealth holdings relative to their peers, so they favor saving over consumption. But in equilibrium relative wealth is fixed at zero since everybody is the same; hence the only way to save more relative to consumption is to reduce consumption. In normal times the central bank would offset this reduction in aggregate demand by reducing the nominal interest rate. This is not an option at the ZLB, so output falls.
Figure 8. WUNK model: other properties at the ZLB

\( \pi \) is inflation; \( y \) is output; \( y^o \) is natural output; \( c \) is private consumption; \( c^o \) is natural consumption; the Phillips line represents the steady-state Phillips curve (9) in panels A–C, and the steady-state Phillips curve (27) in panel D; the Euler line represents the steady-state Euler equation (29) in panels A–C and (26) in panel D. The ZLB equilibrium is at the intersection of the Phillips and Euler lines: output/consumption is below its natural level and inflation is negative. Panel A illustrates the paradox of thrift: increasing the marginal utility of wealth steepens the Euler line, which depresses output and inflation without changing relative wealth. Panel B illustrates the paradox of toil: reducing the disutility of labor moves the Phillips line outward, which depresses output, inflation, and hours worked. Panel C illustrates the paradox of flexibility: decreasing the price-adjustment cost rotates the Phillips line counterclockwise around the natural steady state, which depresses output and inflation. Panel D shows that the government-spending multiplier is above one: increasing government spending shifts the Phillips line upward, which raises private consumption and therefore increases output more than one-for-one.
4.2. Paradox of toil

Next we consider a reduction in the disutility of labor ($\kappa$). In this case the steady-state Phillips curve changes while the steady-state Euler equation does not. Using (8), we rewrite the steady-state Phillips curve (9):

$$\pi = \frac{\varepsilon \kappa}{\delta y a} \gamma - \frac{\varepsilon - 1}{\delta y}.$$

Hence reducing the disutility of labor flattens the Phillips line, which moves the economy inward along the Euler line: both output and inflation decrease (figure 8, panel B). Moreover, since hours worked and output are related by $h = y/a$, hours fall as well. The following proposition states the result:

**Proposition 7.** At the ZLB in the WUNK model, the paradox of toil holds: an unexpected and permanent reduction in the disutility of labor reduces hours worked, output, and inflation.

The paradox of toil was discovered by Eggertsson (2010, p. 15) and Eggertsson and Krugman (2012, p. 1487). It operates as follows. With lower disutility of labor, real marginal costs are lower, and the natural level of output is higher: firms would like to produce and sell more. To increase sales, firms tend to reduce their prices, reducing inflation. Away from the ZLB, the central bank would offset this reduction in inflation by lowering the nominal interest rate. But this cannot happen at the ZLB, so the reduction in inflation raises the real interest rate—as the nominal interest rate is at zero—which pushes households to save more. In equilibrium, this lowers output and hours worked.

As usual in this context, an increase in technology ($a$) would have the same effect as a reduction in the disutility of labor: it would lower output and inflation.

4.3. Paradox of flexibility

We then examine a decrease in the price-adjustment cost ($\gamma$). The steady-state Euler equation is not affected, but the steady-state Phillips curve is. Equation (9) shows that decreasing the price-adjustment cost leads to a counterclockwise rotation of the Phillips line around natural steady state, which moves the economy inward along the Euler line: both output and inflation decrease (figure 8, panel C). Proposition 8 records the results:

**Proposition 8.** At the ZLB in the WUNK model, the paradox of flexibility holds: an unexpected and permanent decrease in price-adjustment cost reduces output and inflation.

The paradox of flexibility was discovered by Werning (2011, pp. 13–14) and Eggertsson and Krugman (2012, pp. 1487–1488). Intuitively, with a lower price-adjustment cost, firms are keener
to adjust their prices to bring production closer to the natural level of output, which accentuates the existing deflation. At the ZLB, lower inflation means higher real interest rate, which makes households more prone to save and less to consume. In equilibrium, this reduces output.

4.4. Above-one government-spending multiplier

We finally look at an increase in government spending \((g)\), using the model with government spending introduced in section 3.3. From (27) we see that increasing government spending shifts the Phillips line upward, which moves the economy upward along the Euler line: both private consumption and inflation increase (figure 8, panel D). Since private consumption increases when public consumption does, the government-spending multiplier \(\frac{dy}{dg} = 1 + \frac{dc}{dg}\) is greater than one. Proposition 9 gives the results:

**Proposition 9.** At the ZLB in the WUNK model, an unexpected and permanent increase in government spending raises private consumption and inflation. Hence the government-spending multiplier \(\frac{dy}{dg}\) is above one; its value is given by (28).

The multiplier value (28) is derived in appendix E. Christiano, Eichenbaum, and Rebelo (2011), Eggertsson (2011), and Woodford (2011) first showed that at the ZLB in the New Keynesian model, the government-spending multiplier is above one. The intuition is the following. With higher government spending, real marginal costs for a given level of private consumption are higher, so firms would like to reduce their sales to households. Hence, firms tend to increase their prices, raising inflation. At the ZLB, the increase in inflation lowers the real interest rate—as the nominal interest rate is at zero—which makes households more prone to consume. In equilibrium this leads to higher private consumption and a multiplier above one.

5. Empirical assessment of the WUNK assumption

In the WUNK model the marginal utility of wealth is assumed to be above the threshold specified in (13). We now use empirical evidence to assess this assumption.

5.1. Formulation of the WUNK assumption in terms of estimable statistics

As a first step, we re-express the WUNK assumption in terms of estimable statistics. Multiplying (13) by \(\delta y^n\), we obtain

\[
\delta \times u'(0)y^n > y^n\epsilon_k
\frac{y^n}{\gamma a}.
\]
We begin by working on the marginal rate of substitution between wealth and consumption, \( u'(0)y^n \). The definition of the natural rate of interest, given by (11), implies that the marginal rate of substitution satisfies \( u'(0)y^n = \delta - r^n \). This shows how to measure the marginal rate of substitution: by estimating the gap between time discount rate and natural rate of interest.

Second, we work on \( (y^n\epsilon\kappa)/(\gamma a) \). Combining the first-order approximation \( \pi(t) = \pi(t + dt) - \dot{\pi}(t + dt)dt \) and the Phillips curve given by (7), we obtain the following equation:

\[
\pi(t) = \pi(t + dt) - \delta \pi(t + dt)dt + \frac{y^n\epsilon\kappa}{\gamma a} \cdot \frac{y(t) - y^n}{y^n}dt.
\]

Setting the unit of time to one quarter and \( dt = \frac{1}{4} \), we find that in our model, \( \lambda \) can be measured as the coefficient on the output gap in the Phillips curve.

To sum up, we rewrite (13) as

\[
\delta \times (\delta - r^n) > \lambda.
\]

We now survey the empirical literature to obtain estimates of the three statistics required to assess the WUNK assumption: the natural rate of interest \( r^n \), the output-gap coefficient \( \lambda \) in the New Keynesian Phillips curve, and the time discount rate \( \delta \).

5.2. Natural rate of interest

A large number of macroeconometric studies have estimated the natural rate of interest, using different statistical models, methodologies, and data. Recent US studies obtain comparable estimates of \( \lambda \) and \( \delta \), but the results are inconsistent with the assumption of \( \delta \times (\delta - r^n) > \lambda \). This can be established by following the previous steps but using (25) instead of (7) to measure \( \dot{\pi}(t + dt) \). In sum, evaluating (31) is the correct way to assess the WUNK assumption irrespective of the value of the Frisch elasticity.
mates of the natural rate, around 2% per annum on average over the 1985–2015 period (Williams 2017, fig. 1). Accordingly, we use $r^n = 2\%$ as our estimate.

5.3. Output-gap coefficient in the New Keynesian Phillips curve

A large literature has estimated New Keynesian Phillips curves. Mavroeidis, Plagborg-Moller, and Stock (2014, sec. 5) offers a synthesis by generating estimates of the New Keynesian Phillips curve using an array of data, methods, and specifications from the literature. There is significant uncertainty around the estimation, but in many cases the output-gap coefficient is positive and very small. Overall, their median estimate of the output-gap coefficient is $\lambda = 0.004$ (table 5, row 1), which we use as our estimate.

5.4. Time discount rate

*Literature surveys.* Since the 1970s many studies have estimated time discount rates, using field and laboratory experiments and real-world behavior. Frederick, Loewenstein, and O’Donoghue (2002, table 1) survey 43 such studies. The estimates are quite dispersed, but the majority of them points to high time discount rates, much higher than prevailing market interest rates. We compute the mean estimate in each of the studies covered by the survey, and then compute the median value of these means. We obtain an annual discount rate of $\delta = 35\%$.

There is one immediate limitation with the studies discussed by Frederick, Loewenstein, and O’Donoghue: they use a single rate to exponentially discount future utility. But exponential discounting does not describe reality well because people seem to choose more impatiently for the present than for the future—they exhibit present-focused preferences (Ericson and Laibson 2019). Recent studies have moved away from exponential discounting and allowed for present-focused preferences, including quasi-hyperbolic ($\beta-\delta$) discounting. Andersen et al. (2014, table 3) survey 16 such studies, concentrating on experimental studies with real incentives. We compute again the mean estimate in each study, and then the median value of these means. We obtain an annual time discount rate of $\delta = 43\%$; therefore, discount rates remain high even after accounting for present-focus.

*Other potential issues.* There are two potential issues with the studies in Andersen et al. (2014)—which could explain why they find such high discount rates. First, many of the studies are run with university students instead of subjects representative of the general population. There does not seem to be systematic differences in discounting between student and non-student
subjects, however (Cohen et al. 2019, sec. 6A). Hence, using students subjects is unlikely to bias the estimates reported by Andersen et al.

Second, the discount rates in Andersen et al. are elicited from experiments using financial flows, not consumption flows. As the goal is to elicit the time discount rate on consumption, this could be problematic (Cohen et al. 2019, sec. 4B). The problems could be exacerbated if subjects derive utility from wealth. To assess this potential issue, suppose first (as in most of the literature) that monetary payments are consumed at the time of receipt, and that the utility function is locally linear. Under these two conditions, the monetary experiments surveyed by Andersen et al. deliver estimates of the relevant discount rate (Cohen et al. 2019, sec. 4B). If these conditions do not hold, it is more difficult to interpret the experimental findings. For instance, if subjects optimally smooth their consumption over time by borrowing and saving at some market interest rate, then experiments with financial flows only elicit the interest rate faced by subjects, and reveal nothing about their time discount rate (Cohen et al. 2019, sec. 4B). In that case, we should rely on experiments using time-dated consumption rewards instead of monetary rewards. Such experiments directly deliver estimates of the time discount rate. Many such experiments have been conducted; a robust finding is that discount rates for consumption rewards are systematically higher than discount rates for monetary rewards (Cohen et al. 2019, sec. 3A). Hence, the estimates presented in Andersen et al. are, if anything, lower bounds on actual time discount rates.

Summary. We use an annual time discount rate of $\delta = 43\%$, which is a mid-range estimate and takes into account present-focused preferences. Although the estimated financial return on wealth ($r^n = 2\%$) is much lower than the estimated time discount rate ($\delta = 43\%$), people are willing to hold wealth because they derive direct utility from it.

5.5. Assessment

We now combine our empirical estimates of $r^n$, $\delta$, and $\lambda$. Since $\lambda$ is estimated using quarter as a unit of time, we need to express $r^n$ and $\delta$ as quarterly rates: $r^n = 2\%/4 = 0.5\%$ per quarter, and $\delta = 43\%/4 = 10.8\%$ per quarter. Using these estimates, we find that (31) comfortably holds:

\[
\delta \times (\delta - r^n) = 0.108 \times (0.108 - 0.005) = 0.011 > 0.004 = \lambda.
\]

Hence the WUNK assumption holds in US data.

Because the time discount rate is not typically calibrated using microevidence in macroeconomic models, the estimate of $\delta$ used here could appear surprisingly high. But the WUNK
assumption also holds with lower time discount rates. Indeed, (31) holds for any annual time discount rate above 27%, since \((0.27/4) \times [(0.27/4) - 0.005] = 0.0042 > 0.004\). This time discount rate is at the low end of available estimates: 13 of the 16 studies in Andersen et al. (2014, table 3) estimate a mean annual time discount rate above 27%; and in fact in 11 of the 16 studies, the bottom of the range of reported estimates is above 27%.

6. Conclusion

This paper extends the New Keynesian model by introducing relative wealth into the utility function. The marginal utility of wealth is assumed to be above a threshold that depends on price rigidity—an assumption that generally holds in the data. Although our model deviates only minimally from the New Keynesian model, it resolves all the New Keynesian anomalies at the ZLB: even when the ZLB is arbitrarily long-lasting, there is no collapse of inflation and output; and both forward guidance and government spending have limited, plausible effects. At the same time, our model retains other properties of the New Keynesian model at the ZLB: paradox of thrift, paradox of toil, paradox of flexibility, and above-one government-spending multiplier.

Beyond the New Keynesian model, the wealth-in-the-utility assumption might be a simple way to better model people’s saving behavior. First, it reconciles the single-digit interest rates observed on many markets with the double-digit time discount rates measured in most experimental studies. Relatedly, it explains why people have a higher time discount rate for consumption rewards (such as food and beverage) than for monetary rewards. Second, it allows for a broad range of values for the natural rate of interest and steady-state real interest rate—including negative ones. Third, it reduces the effect of future interest rates on today’s consumption: the Euler equation features discounting. And fourth, it leads to a negative steady-state relationship between consumption and real interest rate, as in the old-fashioned IS curve. These properties contrast with the standard model, in which the natural rate of interest and steady-state real interest rate equal the time discount rate, and future interest rates have implausibly large effects on today’s consumption.

References


Appendix A. Derivation of Euler equation and Phillips curve

We derive the two differential equations that describe the equilibrium of our New Keynesian model: the Phillips curve, given by (7); and the Euler equation, given by (10).

Household saving and pricing

We begin by characterizing household $j$’s saving and pricing.

Hamiltonian. The current-value Hamiltonian of the household’s problem is

$$
\mathcal{H}_j = \frac{\varepsilon}{\varepsilon - 1} \ln \left( \int_0^1 c_{jk}(t)^{(e-1)/\varepsilon} \, dk \right) + u \left( \frac{b_j(t) - b(t)}{p(t)} \right) - \frac{K}{a} \frac{y_j^d(p_j(t), t)}{p(t)} - \frac{Y}{2} \pi_j(t)^2
$$

$$
+ \mathcal{A}_j(t) \left[ i(t)b_j(t) + p_j(t)y_j^d(p_j(t), t) - \int_0^1 p_k(t)c_{jk}(t) \, dk - \tau(t) \right] + \mathcal{B}_j(t)\pi_j(t)p_j(t),
$$

with control variables $c_{jk}(t)$ for all $k \in [0, 1]$ and $\pi_j(t)$, state variables $b_j(t)$ and $p_j(t)$, and costate variables $\mathcal{A}_j(t)$ and $\mathcal{B}_j(t)$. Note that we have used the production and demand constraints to substitute $y_j(t)$ and $h_j(t)$ out of the Hamiltonian.

The necessary conditions for a maximum to the household’s problem are given by Acemoglu (2009, theorem 7.9); we apply them here. These conditions form the basis for the model’s equilibrium conditions. (To ease notation we drop the time index $t$.)

Consumption. The first set of optimality conditions are $\partial \mathcal{H}_j / \partial c_{jk} = 0$ for all $k \in [0, 1]$. They yield

(A1) $$
\frac{1}{c_j} \left( \frac{c_{jk}}{c_j} \right)^{-1/\varepsilon} = \mathcal{A}_j p_k.
$$

Appropriately integrating (A1) over all $k \in [0, 1]$, and using (2) and (3), we find

(A2) $$
\mathcal{A}_j = \frac{1}{pc_j}.
$$

Combining (A1) and (A2), we then obtain

(A3) $$
c_{jk} = \left( \frac{p_k}{p} \right)^{-\varepsilon} c_j.
$$
Integrating (A3) over all $j \in [0, 1]$, we get the usual demand for good $k$:

\[(A4)\]

\[
y_k^d(p_k) = \int_0^1 c_{jk} \, dj = \left(\frac{p_k}{p}\right)^{-\epsilon} c,
\]

where $c = \int_0^1 c_j \, dj$ is aggregate consumption. We use this expression for $y_k^d(p_k)$ in household $k$'s Hamiltonian. We also obtain $\int_0^1 p_k c_{jk} \, dk = pc_j$: the price of one unit of consumption index is $p$.

**Bond holdings.** The second optimality condition is $\partial H_j / \partial b_j = \delta A_j - \hat{A}_j$, which gives

\[
-\frac{\hat{A}_j}{A_j} = i + \frac{1}{p A_j} \cdot u'\left(\frac{b_j - b}{p}\right) - \delta.
\]

Using (A2), we obtain the household's Euler equation:

\[(A5)\]

\[
\frac{\dot{c}_j}{c_j} = i - \pi + c_j u'\left(\frac{b_j - b}{p}\right) - \delta.
\]

This Euler equation describes the optimal path for the household's consumption.

**Inflation.** The third optimality condition is $\partial H_j / \partial \pi_j = 0$, which yields

\[(A6)\]

\[
\mathcal{B}_j p_j = \gamma \pi_j.
\]

Differentiating (A6) with respect to time, we obtain

\[(A7)\]

\[
\frac{\dot{\mathcal{B}}_j}{\mathcal{B}_j} = \frac{\dot{\pi}_j}{\pi_j} - \pi_j.
\]

**Price.** The fourth and last optimality condition is $\partial H_j / \partial p_j = \delta \mathcal{B}_j - \dot{\mathcal{B}}_j$, which implies

\[
\frac{\kappa}{a} \cdot \frac{\epsilon y_j}{p_j} - (\epsilon - 1) A_j y_j + \mathcal{B}_j \pi_j = \delta \mathcal{B}_j - \dot{\mathcal{B}}_j.
\]

Reshuffling the terms, we obtain

\[
\pi_j - \frac{(\epsilon - 1) y_j A_j}{\mathcal{B}_j p_j} \left( p_j - \frac{\epsilon}{\epsilon - 1} \cdot \frac{\kappa}{a A_j} \right) = \delta - \frac{\dot{\mathcal{B}}_j}{\mathcal{B}_j}.
\]
Then, using (A2), (A6), and (A7), we obtain the household’s Phillips curve:

\[
\frac{\dot{\pi}_j}{\pi_j} = \delta + \left( \frac{\epsilon - 1}{\gamma c_j\pi_j} \right) \left( \frac{p_j - \epsilon}{\epsilon - 1} \cdot \frac{\kappa c_j}{a} \right). \tag{A8}
\]

This equation describes the optimal path for the price set by the household.

**Equilibrium saving and pricing**

We turn to saving and pricing in equilibrium. All households have the same initial wealth and initial price, so they all behave the same. We therefore omit the subscripts \( j \) and \( k \).

Then, we simplify the household’s Euler equation, given by (A5), and the household’s Phillips curve, given by (A8), using two equilibrium conditions: relative wealth is zero \( (b_j = b) \); and production and consumption are equal \( (y = c) \). Accordingly, we simplify the household’s Euler equation to

\[
\frac{\dot{y}}{y} = r - \delta + \nu'(0)y,
\]

where \( r = i - \pi \). This is just the Euler equation (10).

Since \( y^n = (\epsilon - 1)a/(\epsilon \kappa) \), we also simplify the household’s Phillips curve to

\[
\dot{\pi} = \delta \pi - \frac{\epsilon \kappa}{\gamma a} (y - y^n).
\]

This is the Phillips curve (7).

**Appendix B. Euler equation and Phillips curve in discrete time**

We recast the model of section 2 in discrete time, and we rederive the Euler equation and Phillips curve. This reformulation is helpful to compare our model to the textbook New Keynesian model, which is presented in discrete time (see Woodford 2003; Gali 2008). The reformulation also shows that introducing wealth in the utility function yields a discounted Euler equation.

**Assumptions**

The discrete-time model is the same as the continuous-time model, except for government bonds. In discrete time, households trade one-period government bonds. Bonds purchased in period \( t \) have a price \( q(t) \) and pay one unit of money at maturity, in period \( t + 1 \). The nominal interest rate between \( t \) and \( t + 1 \) is defined as \( i(t) = -\ln(q(t)) \).
Household saving and pricing

We begin by characterizing household $j$’s saving and pricing.

**Household problem.** The household chooses sequences $\{y_j(t), p_j(t), h_j(t), [c_{jk}(t)]_{k=0}^1, b_j(t)\}_{t=0}^\infty$ to maximize the discounted sum of instantaneous utilities

$$\sum_{t=0}^{\infty} \beta^t \left( \frac{\epsilon}{\epsilon - 1} \ln \left( \int_0^1 c_{jk}(t)^{(\epsilon-1)/\epsilon} \, dk \right) + u \left( \frac{b_j(t) - b(t)}{p(t)} \right) - \kappa h_j(t) - \frac{\gamma}{2} \left[ \frac{p_j(t)}{p_j(t-1)} - 1 \right]^2 \right) \, dt,$$

where $\beta < 1$ is the time discount factor. The maximization is subject to three constraints. First, there is the production function (1). Second, there is the demand for good $j$, given by (6). The demand for good $j$ is the same as in continuous time because the allocation of consumption expenditure across goods is a static decision, so it is unaffected by the representation of time. And third, there is a budget constraint:

$$\int_0^1 p_k(t)c_{jk}(t) \, dk + q(t)b_j(t) + \tau(t) = p_j(t)y_j(t) + b_j(t - 1).$$

Household $j$ is also subject to a solvency constraint preventing Ponzi schemes: $\lim_{T \to \infty} b_j(T) \geq 0$. Lastly, household $j$ takes as given the initial conditions $b_j(-1)$ and $p_j(-1)$, as well as the sequences of aggregate variables $\{p(t), q(t), c(t)\}_{t=0}^\infty$.

**Lagrangian.** The Lagrangian of the household’s problem is

$$L_j = \sum_{t=0}^{\infty} \beta^t \left( \frac{\epsilon}{\epsilon - 1} \ln \left( \int_0^1 c_{jk}(t)^{(\epsilon-1)/\epsilon} \, dk \right) + u \left( \frac{b_j(t) - b(t)}{p(t)} \right) - \kappa y_j^d[p_j(t), t] - \gamma \left[ \frac{p_j(t)}{p_j(t-1)} - 1 \right]^2 \right)$$

$$+ A_j(t) \left[ p_j(t)b_j(t) + b_j(t - 1) - \int_0^1 p_k(t)c_{jk}(t) \, dk - q(t)b_j(t) - \tau(t) \right]$$

where $A_j(t)$ is the Lagrange multiplier on the budget constraint in period $t$, and

$$(A9) \quad y_j^d[p_j(t), t] = \left[ \frac{p_j(t)}{p(t)} \right]^{-\epsilon} c(t)$$

is the demand for good $j$ in period $t$. Note that we have used the production and demand constraints to substitute $y_j(t)$ and $h_j(t)$ out of the Lagrangian.

The necessary conditions for a maximum to the household’s problem are standard first-order conditions: with respect to $c_{jk}(t)$ for all $k \in [0, 1]$, $b_j(t)$, and $p_j(t)$, for all $t \geq 0$. 

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Consumption. The first set of first-order conditions is \( \partial \mathcal{L}_j / \partial c_{jk}(t) = 0 \) for all \( k \in [0, 1] \) and all \( t \).

As in continuous time we obtain

\[
A_j(t) = \frac{1}{p(t)c_j(t)}.
\]

Bond holdings. The second first-order condition is \( \partial \mathcal{L}_j / \partial b_j(t) = 0 \) for all \( t \), which gives

\[
q(t)A_j(t) = \frac{1}{p(t)} u'(\frac{b_j(t) - b(t)}{p(t)}) + \beta A_j(t + 1).
\]

Using (A10), we obtain the household's Euler equation for consumption:

\[
q(t) = c_j(t)u'(\frac{b_j(t) - b(t)}{p(t)}) + \beta p(t)c_j(t)
\]

Price. The third first-order condition is \( \partial \mathcal{L}_j / \partial p_j(t) = 0 \) for all \( t \), which yields

\[
0 = \frac{\kappa}{a} \cdot \epsilon y_j(t) - \frac{\gamma}{p_j(t)} \left[ \frac{p_j(t)}{p_j(t - 1)} - 1 \right] + (1 - \epsilon)A_j(t)y_j(t) + \beta \frac{p_j(t + 1)}{p_j(t)} \left[ \frac{p_j(t + 1)}{p_j(t)} - 1 \right].
\]

Multiplying this equation by \( p_j(t)/\gamma \) and using (A10), we obtain the household's Phillips curve:

\[
\frac{p_j(t)}{p_j(t - 1)} \left[ \frac{p_j(t)}{p_j(t - 1)} - 1 \right] = \beta \frac{p_j(t + 1)}{p_j(t)} \left[ \frac{p_j(t + 1)}{p_j(t)} - 1 \right] + \frac{\epsilon \kappa}{\gamma} y_j(t) - \frac{\epsilon - 1}{\gamma} \cdot \frac{p_j(t)y_j(t)}{p(t)c_j(t)}.
\]

Equilibrium saving and pricing

We turn to saving and pricing in equilibrium. All households behave the same in equilibrium, so we drop the subscripts \( j \) and \( k \). Particularly, all households hold the same wealth, so relative wealth is zero: \( b_j(t) = b(t) \). As production and consumption are equal, we set \( y(t) = c(t) \). Then, from (A11), we obtain the Euler equation:

\[
q(t) = u'(0)y(t) + \beta \frac{p(t)y(t)}{p(t + 1)y(t + 1)}.
\]

Similarly, using (A12), and using (8), we obtain the Phillips curve:

\[
\frac{p(t)}{p(t - 1)} \left[ \frac{p(t)}{p(t - 1)} - 1 \right] = \beta \frac{p(t + 1)}{p(t)} \left[ \frac{p(t + 1)}{p(t)} - 1 \right] + \frac{\epsilon - 1}{\gamma} \left[ \frac{y(t)}{y^n} - 1 \right].
\]
Log-linearization

Last, to obtain standard expressions, we log-linearize the Euler equation and Phillips curve around the natural steady state—where $y = y^n$, $\pi = 0$, and $i = r^n$. We introduce the log-deviation of output from its steady-state level: $\hat{y}(t) = \ln(y(t)) - \ln(y^n)$. We also introduce the inflation rate between periods $t$ and $t + 1$: $\pi(t + 1) = \ln(p(t + 1)) - \ln(p(t))$.

Euler equation. We start by log-linearizing the Euler equation (A/one.lf/three.lf). We first take the log of the left-hand side of (A/one.lf/three.lf). Using the discrete-time definition of the interest rate $i(t)$, we obtain $\ln(q(t)) = -i(t)$.

Next we take the log of the right-hand side of (A/one.lf/three.lf) and obtain $\Lambda \equiv \ln(\Lambda_{1} + \Lambda_{2})$, where

$$\Lambda_{1} \equiv u'(0)y(t)$$
$$\Lambda_{2} \equiv \beta \frac{p(t)y(t)}{p(t + 1)y(t + 1)}.$$

For future reference, we compute the values of $\Lambda$, $\Lambda_{1}$, and $\Lambda_{2}$ at the natural steady state. At the natural steady state, $i = r^n$, so the log of the left-hand side of (A/one.lf/three.lf) equals $-r^n$, which implies that the log of the right-hand side of (A/one.lf/three.lf) must also equal $-r^n$—thus $\Lambda = -r^n$. Moreover, at the natural steady state, $\Lambda_{1} = u'(0)y^n$. And, since inflation is zero and output is constant at that steady state, $\Lambda_{2} = \beta$.

Using the results, we obtain a first-order approximation of $\Lambda(\Lambda_{1}, \Lambda_{2})$ around the natural steady state:

$$\Lambda = -r^n + u'(0)y^n \cdot \frac{\partial \Lambda}{\partial \Lambda_{1}} \cdot \left[\frac{y(t)}{y^n} - 1\right] + \beta \cdot \frac{\partial \Lambda}{\partial \Lambda_{2}} \cdot \left[\frac{p(t)y(t)}{p(t + 1)y(t + 1)} - 1\right].$$

Factoring out $u'(0)y^n$ and $\beta$, and using the definitions of $\Lambda_{1}$ and $\Lambda_{2}$, we obtain

(A15) $\Lambda = -r^n + u'(0)y^n \cdot \frac{\partial \Lambda}{\partial \Lambda_{1}} \cdot \left[\frac{y(t)}{y^n} - 1\right] + \beta \cdot \frac{\partial \Lambda}{\partial \Lambda_{2}} \cdot \left[\frac{p(t)y(t)}{p(t + 1)y(t + 1)} - 1\right].$

Since $\Lambda = \ln(\Lambda_{1} + \Lambda_{2})$, we obviously have

$$\frac{\partial \Lambda}{\partial \Lambda_{1}} = \frac{\partial \Lambda}{\partial \Lambda_{2}} = \frac{1}{\Lambda_{1} + \Lambda_{2}}.$$

In the first-order approximation, the partial derivatives are evaluated at the natural state, so their value is

$$\frac{\partial \Lambda}{\partial \Lambda_{1}} = \frac{\partial \Lambda}{\partial \Lambda_{2}} = \frac{1}{u'(0)y^n + \beta}.$$
Hence, (A/one.lf/five.lf) becomes

\[ \Lambda = -r^n + \frac{u'(0)y^n}{u'(0)y^n + \beta} \left[ y(t) - y^n - 1 \right] + \frac{\beta}{u'(0)y^n + \beta} \left[ \frac{p(t)y(t)}{p(t + 1)y(t + 1)} - 1 \right]. \]

The last step is to note that the first-order approximation of \( \ln(x) \) at \( x = 1 \) is \( x - 1 \), so that around the natural steady state, and up to second-order terms, we have

\[ \frac{y(t)}{y^n} - 1 = \ln \left( \frac{y(t)}{y^n} \right) = \hat{y}(t) \]

and

\[ \frac{p(t)y(t)}{p(t + 1)y(t + 1)} - 1 = \ln \left( \frac{p(t)y(t)}{p(t + 1)y(t + 1)} \right) = \ln \left( \frac{y(t)}{y^n} \right) - \ln \left( \frac{y(t + 1)}{y^n} \right) - \ln \left( \frac{p(t + 1)}{p(t)} \right) = \hat{y}(t) - \hat{y}(t + 1) - \pi(t + 1). \]

Hence, we rewrite (A6) as

\[ \Lambda = -r^n + \frac{u'(0)y^n}{u'(0)y^n + \beta} \hat{y}(t) + \frac{\beta}{u'(0)y^n + \beta} [\hat{y}(t) - \hat{y}(t + 1) - \pi(t + 1)] \]

\[ = -r^n + (1 - \alpha)\hat{y}(t) + \alpha [\hat{y}(t) - \hat{y}(t + 1) - \pi(t + 1)] \]

where

\[ \alpha = \frac{\beta}{\beta + u'(0)y^n}. \]

In conclusion, taking the log of the Euler equation (A13) yields

\[ -i(t) = -r^n + (1 - \alpha)\hat{y}(t) + \alpha [\hat{y}(t) - \hat{y}(t + 1) - \pi(t + 1)] \]

This equation is valid up to terms that are second order around the natural steady state. Reshuffling the terms yields

\[ (A18) \]

\[ \hat{y}(t) = \alpha \hat{y}(t + 1) - [i(t) - r^n - \alpha \pi(t + 1)]. \]

**Discounting.** Because the marginal utility of wealth \( u'(0) \) is positive, we have \( \alpha = \beta / [\beta + u'(0)y^n] < 1 \) in (A18). Thus the Euler equation is discounted: future output, \( \hat{y}(t + 1) \), appears discounted by the coefficient \( \alpha < 1 \). Such discounting may appear for a variety of reasons: overlapping generations.
(Del Negro, Giannoni, and Patterson 2015; Eggertsson, Mehrotra, and Robbins 2019); heterogeneous agents facing borrowing constraints and cyclical income risk (McKay, Nakamura, and Steinsson 2017; Acharya and Dogra 2019; Bilbiie 2019); consumers’ bounded rationality (Gabaix 2016); incomplete information (Angeletos and Lian 2018); bonds in the utility function (Campbell et al. 2017); and a cost of borrowing increasing in household debt (Beaudry and Portier 2018).

To make discounting more apparent, we solve the Euler equation forward:

$$\hat{y}(t) = -\sum_{k=0}^{+\infty} \alpha^k \left[ i(t+k) - r^n - \alpha \pi(t+k+1) \right].$$

The effect on current output of interest rates $k$ periods in the future is discounted by $\alpha^k < 1$; hence, discounting is stronger for interest rates further in the future (McKay, Nakamura, and Steinsson 2017, p. 821).

**Phillips curve.** Next we log-linearize the Phillips curve (A14). We start with the left-hand side of (A14). Note that the first-order approximations of $x(x-1)$ and $\ln(x)$ at $x = 1$ are both $x - 1$. This means that up to second-order terms around $x = 1$, we have $x(x-1) = \ln(x)$. Hence, up to second-order terms around the natural steady state,

$$\frac{p(t)}{p(t-1)} \left[ \frac{p(t)}{p(t-1)} - 1 \right] = \ln \left( \frac{p(t)}{p(t-1)} \right) = \pi(t).$$

We turn to the right-hand side of (A14). Following the same logic, up to second-order terms around the natural steady state, we have

$$\beta \frac{p(t+1)}{p(t)} \left[ \frac{p(t+1)}{p(t)} - 1 \right] = \beta \ln \left( \frac{p(t+1)}{p(t)} \right) = \beta \pi(t+1).$$

Furthermore, using (A17), we know that up to second-order terms around the natural steady state, we have

$$\frac{\epsilon - 1}{\gamma} \left[ \frac{y(t)}{y^n} - 1 \right] = \frac{\epsilon - 1}{\gamma} \hat{y}(t).$$

Combining all these results, we find that the Phillips curve (A14) implies

$$\pi(t) = \beta \pi(t+1) + \frac{\epsilon - 1}{\gamma} \hat{y}(t).$$

This equation is valid up to terms that are second order around the natural steady state.
Appendix C. Proofs

We provide alternative proofs of propositions 1 and 2. These proofs are not based on phase diagrams but instead are algebraic; they are closer to the proofs found in the literature. We also complement the graphical proof of proposition 4 presented in the main text.

Alternative proof of proposition 1

**Steady state.** A steady state must satisfy the steady-state Phillips curve (9) and the steady-state Euler equation (12), where monetary policy imposes \( r(\pi) = r^n + (\phi - 1)\pi \). These equations form a linear system:

\[
\pi = \frac{\epsilon K}{\delta Ya} (y - y^n)
\]

\[
(\phi - 1)\pi = -u'(0)(y - y^n).
\]

As \([y = y^n, \pi = 0]\) satisfies both equations, it is a steady state. Furthermore the two equations are non-parallel. In the NK model this is obvious since \( u'(0) = 0 \). In the WUNK model the slope of the second equation is \(-u'(0)/(\phi - 1)\); if \( \phi > 1 \), the slope is negative; if \( \phi \in [0, 1) \), the slope is positive and greater than \( u'(0) \) and thus than \( \epsilon K/(\delta Ya) \), as (13) holds; in both cases the two equations have different slope. We conclude that \([y^n, 0]\) is the unique steady state. The nominal interest rate at \([y^n, 0]\) is given by \( i = r^n + \phi \times 0 = r^n > 0 \).

**Linearization.** Dynamics are governed by the nonlinear dynamical system generated by the Phillips curve (7) and the Euler equation (10), where monetary policy imposes \( r(\pi) = r^n + (\phi - 1)\pi \). The Phillips curve can be written \( \dot{\pi}(t) = P(y(t), \pi(t)) \) where \( P(y, \pi) = \delta\pi - \epsilon K(y - y^n)/(\gamma a) \); the Euler equation (10) can be written \( \dot{y}(t) = E(y(t), \pi(t)) \) where \( E(y, \pi) = y[\phi(\phi - 1)\pi + u'(0)(y - y^n)] \). The dynamical system is linearized around the natural steady state as follows:

\[
\begin{bmatrix}
\dot{y}(t) \\
\dot{\pi}(t)
\end{bmatrix} =
\begin{bmatrix}
\frac{\partial E}{\partial y} & \frac{\partial E}{\partial \pi} \\
\frac{\partial P}{\partial y} & \frac{\partial P}{\partial \pi}
\end{bmatrix}
\begin{bmatrix}
y(t) - y^n \\
\pi
\end{bmatrix},
\]

where the partial derivatives are evaluated at \([y = y^n, \pi = 0]\). We have \( \partial E/\partial y = y^n u'(0) \), \( \partial E/\partial \pi = y^n(\phi - 1) \), \( \partial P/\partial y = -\epsilon K/(\gamma a) \), and \( \partial P/\partial \pi = \delta \). Hence this linearized system is just (15).

Around the natural steady state, equilibrium trajectories satisfy the linear dynamical system (15). To study the system, we denote by \( M \) the matrix in (15), and by \( \mu_1 \in \mathbb{C} \) and \( \mu_2 \in \mathbb{C} \) the two eigenvalues of \( M \), assumed to be distinct.
**Solution with two real eigenvalues.** We begin by solving (15) when $\mu_1$ and $\mu_2$ are real and nonzero. Without loss of generality, we assume $\mu_1 < \mu_2$. Then the solution to (15) takes the form

(A20) \[
\begin{bmatrix}
y(t) - y^a \\
\pi(t)
\end{bmatrix} = x_1 e^{\mu_1 t} \mathbf{v}_1 + x_2 e^{\mu_2 t} \mathbf{v}_2,
\]

where $\mathbf{v}_1 \in \mathbb{R}^2$ and $\mathbf{v}_2 \in \mathbb{R}^2$ are the linearly independent eigenvectors respectively associated with the eigenvalues $\mu_1$ and $\mu_2$, and $x_1 \in \mathbb{R}$ and $x_2 \in \mathbb{R}$ are constants determined by the terminal condition (Hirsch, Smale, and Devaney 2013, p. 35).

From (A20), we see that the system is a source around the steady state if $\mu_1 > 0$ and $\mu_2 > 0$. Moreover, the trajectories are tangent to $\mathbf{v}_1$ when $t \to -\infty$ and are parallel to $\mathbf{v}_2$ when $t \to +\infty$. The system is a saddle if $\mu_1 < 0$ and $\mu_2 > 0$; in that case, the vector $\mathbf{v}_1$ gives the direction of the stable line (saddle path) while the vector $\mathbf{v}_2$ gives the direction of the unstable line. Lastly, if $\mu_1 < 0$ and $\mu_2 < 0$, the system is a sink. (See Hirsch, Smale, and Devaney 2013, pp. 40–44.)

**Solution with two complex eigenvalues.** Next we solve (15) when $\mu_1$ and $\mu_2$ are complex conjugates. We write the eigenvalues as $\mu_1 = \theta + i\zeta$ and $\mu_2 = \theta - i\zeta$. We also write the eigenvector associated with $\mu_1$ as $\mathbf{v}_1 + i\mathbf{v}_2$, where the vectors $\mathbf{v}_1 \in \mathbb{R}^2$ and $\mathbf{v}_2 \in \mathbb{R}^2$ are linearly independent. Then the solution to (15) takes a more complicated form:

\[
\begin{bmatrix}
y(t) - y^a \\
\pi(t)
\end{bmatrix} = e^{\theta t} [\mathbf{v}_1, \mathbf{v}_2] \begin{bmatrix} \cos(\zeta t) & \sin(\zeta t) \\ -\sin(\zeta t) & \cos(\zeta t) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix},
\]

where $[\mathbf{v}_1, \mathbf{v}_2] \in \mathbb{R}^{2 \times 2}$ is a $2 \times 2$ matrix whose columns are respectively the real and imaginary components of an eigenvector of $M$, and $x_1 \in \mathbb{R}$ and $x_2 \in \mathbb{R}$ are constants determined by the terminal condition (Hirsch, Smale, and Devaney 2013, pp. 44–55).

These solutions wind periodically around the steady state, either moving toward it ($\theta < 0$) or away from it ($\theta > 0$). Hence, the system is (spiral) source if the $\theta > 0$ and a (spiral) sink if $\theta < 0$. In the special case $\theta = 0$, the solutions just circle the steady state: the system is a center. (See Hirsch, Smale, and Devaney 2013, pp. 44–47.)

**Classification methodology.** We now use the trace and determinant of $M$ to classify the linear dynamical system (15), using the previous results and the properties that $\text{tr}(M) = \mu_1 + \mu_2$ and $\text{det}(M) = \mu_1 \mu_2$. This methodology is standard (see Hirsch, Smale, and Devaney 2013, pp. 61–64). The following situations will occur in the NK and WUNK models:

- $\text{det}(M) < 0$: the system is a saddle. This is because $\text{det}(M) < 0$ indicates that $\mu_1$ and $\mu_2$ are real,
nonzero, and of opposite sign. (Indeed, if $\mu_1$ and $\mu_2$ were real and of the same sign, $\det(M) = \mu_1\mu_2 > 0$; and if they were complex conjugates, $\det(M) = \mu_1\bar{\mu}_1 = \text{Re}(\mu_1)^2 + \text{Im}(\mu_1)^2 > 0$.)

- $\det(M) > 0$ and $\text{tr}(M) > 0$: the system is a source. This is because $\det(M) > 0$ indicates that $\mu_1$ and $\mu_2$ are real, nonzero, and of the same sign; or complex conjugates. Since in addition $\text{tr}(M) = \delta > 0$, $\mu_1$ and $\mu_2$ must be real and positive, or complex with positive real part. (Indeed, if $\mu_1$ and $\mu_2$ were real and negative, $\text{tr}(M) = \mu_1 + \mu_2 < 0$; if they were complex conjugates with negative real part, $\text{tr}(M) = \mu_1 + \bar{\mu}_1 = 2\text{Re}(\mu_1) < 0$.)

Application to NK and WUNK models. We now compute the trace and determinant of $M$ to classify the dynamical systems of the NK and WUNK models. From (15), we obtain

$$\text{tr}(M) = \delta + u'(0)y^n$$
$$\det(M) = \delta u'(0)y^n + (\phi - 1)\frac{\epsilon\kappa}{\gamma a} y^n.$$

In the NK model, $u'(0) = 0$, so $\text{tr}(M) = \delta$ and $\det(M) = (\phi - 1)y^n\epsilon\kappa/(\gamma a)$. If $\phi > 1$, $\det(M) > 0$ and $\text{tr}(M) > 0$, so the system is a source. If $\phi < 1$, $\det(M) < 0$, so the system is a saddle.

In the WUNK model, since $\phi - 1 \geq -1$ and (13), we have

$$\det(M) \geq \delta u'(0)y^n - \frac{\epsilon\kappa}{\gamma a} y^n = \delta y^n \left[ u'(0) - \frac{\epsilon\kappa}{\gamma a} \right] > 0.$$ 

Furthermore, $\text{tr}(M) > \delta > 0$. Since $\det(M) > 0$ and $\text{tr}(M) > 0$, the system is a source.

Alternative proof of proposition 2

Steady state. A steady state must satisfy the steady-state Phillips curve (9) and the steady-state Euler equation (12), where monetary policy imposes $r(\pi) = -\pi$. These equations form a linear system:

$$\pi = \frac{\epsilon\kappa}{\gamma a} (y - y^n)$$

(A21)

$$\pi = -r^n + u'(0)(y - y^n).$$

(A22)

A solution to this system with positive output is a steady state.
In the NK model, \( u'(0) = 0 \), so the system admits a unique solution:

\[
\begin{align*}
\pi^z &= -r^n \\
y^z &= y^n - \frac{\delta y a}{\epsilon \kappa} r^n.
\end{align*}
\]

Since \( r^n < 0 \), the solution satisfies \( y^z > y^n > 0 \): the solution has positive output so it is a steady state. Hence the NK model admits a unique steady state at the ZLB, where \( \pi^z > 0 \) (since \( r^n < 0 \)) and \( y^z > y^n \). Note that the expressions (16) and (17) reduce to (24) and (23) when \( u'(0) = 0 \).

In the WUNK model, since (13) holds, the two equations in the linear system are non-parallel, so the system admits a unique solution. Substituting \( y - y^n \) out of (22) using (21), we find that inflation in that solution is given by (17). Condition (13) implies that \( u'(0) \delta y a / (\epsilon \kappa) > 1 \), so \( \pi^z \) has the sign of \( r^n \); since \( r^n < 0 \), we infer that \( \pi^z < 0 \). Next, using (21) and the value of \( \pi \) given by (17), we find that output in the unique solution is given by (16). Since (13) holds and \( r^n < 0 \), we infer that \( y^z < y^n \). The last step is to verify that \( y^z > 0 \). Using (16), we need

\[
y^n > \frac{-r^n}{u'(0) - \epsilon \kappa / (\delta y a)}.
\]

Since \( -r^n = u'(0) y^n - \delta \) and \( u'(0) - \epsilon \kappa / (\delta y a) > 0 \) (from (11) and (13)), this is equivalent to

\[
\left[ u'(0) - \frac{\epsilon \kappa}{\delta y a} \right] y^n > u'(0) y^n - \delta.
\]

Eliminating \( u'(0) y^n \) on both sides, we find that this is equivalent to

\[
-\frac{\epsilon \kappa y^n}{\delta y a} > -\delta,
\]

or

\[
\frac{\epsilon \kappa y^n}{\gamma a} < \delta^2.
\]

Using (8), we have \( (\epsilon \kappa y^n) / (\gamma a) = (\epsilon - 1) / \gamma \). So we need to verify that \( \delta > \sqrt{(\epsilon - 1) / \gamma} \). But we have imposed \( \delta > \sqrt{(\epsilon - 1) / \gamma} \) in the WUNK model to ensure that the model accommodates positive natural rates of interest. Given this assumption, we conclude that \( y^z > 0 \): the solution to the system has positive output, so it is a steady state. In sum, the WUNK model admits a unique steady state at the ZLB, where \( \pi^z < 0 \) and \( y^z < y^n \).

**Linearization.** Dynamics around the ZLB steady state are described by the system resulting from the linearization of the dynamical system generated by the Phillips curve (7), the Euler
equation (10), and the monetary policy rule \( i(\pi) = 0 \). This linear system is (18). We denote by \( M \) the matrix in (18); it is obtained from the matrix in (15) after setting \( \phi = 0 \) and replacing \( y^n \) by \( y^z \).

**Classification.** We classify the linear system (18) by computing the trace and determinant of \( M \). We have

\[
\text{tr}(M) = \delta + u'(0)y^z > 0 \quad \text{and} \quad \det(M) = \delta y^z \left[ u'(0) - \frac{\epsilon \kappa}{\delta y^a} \right].
\]

In the NK model, \( u'(0) = 0 \) so \( \det(M) < 0 \), which implies that (18) is a saddle. In the WUNK model, (13) implies that \( \det(M) > 0 \); since in addition \( \text{tr}(M) > 0 \), (18) is a source. In fact, in the WUNK model, the discriminant of the characteristic equation of \( M \) is strictly positive:

\[
\text{tr}(M)^2 - 4 \det(M) = \delta^2 + [u'(0)y^n]^2 + 2\delta u'(0)y^n - 4\delta u'(0)y^n + 4\frac{\epsilon \kappa}{\delta y^a}y^n = \left[ \delta - u'(0)y^n \right]^2 + 4\frac{\epsilon \kappa}{\delta y^a}y^n > 0.
\]

Hence the eigenvalues of \( M \) are real, not complex: (18) is a nodal source, not a spiral source.

**Proof of proposition 4: complement**

We characterize the forward-guidance duration \( \Delta^* \) in the NK model and the ZLB duration \( T^* \) in the WUNK model.

In the NK model, the forward-guidance duration \( \Delta^* \) is the duration that brings the economy on the unstable line of the ZLB dynamical system at time \( T \) (panel C of figure 4). With any longer forward guidance, at time \( T \) the economy is above the unstable line, and so it is connected to ZLB trajectories coming from the northeast quadrant of the phase diagram (panel D of figure 4). Then, both during the ZLB and during forward guidance, inflation is positive and output is above its natural level. Moreover, since the position of the economy at the end of the ZLB is unaffected by the duration of the ZLB, continuously increasing the duration of the ZLB when \( \Delta > \Delta^* \) will lead initial output and inflation to be infinitely high.

In the WUNK model, for any forward-guidance duration, the economy at the beginning of forward guidance is bound to be in the right-hand green triangle of figure 5, panel D. All the points in that triangle are connected to ZLB trajectories that flow from the ZLB steady state, through the left-hand green triangle of figure 5, panel D. For any of these trajectories, initial inflation \( \pi(0) \) converges from above to the ZLB steady state’s inflation \( \pi^z \) as the ZLB duration \( T \) goes to infinity. Since \( \pi^z < 0 \), we infer that for each trajectory, there is a \( \hat{T} \), such that for any \( T > \hat{T} \), \( \pi(0) < 0 \). (Furthermore, as showed in panel D of figure 5, \( y(0) < y^n \) whenever \( \pi(0) < 0 \).) Then we have \( T^* = \max \{ \hat{T} \} \). The maximum exists because the right-hand green triangle is a closed
and bounded subset of $\mathbb{R}^2$, so the set $\{\hat{T}\}$ is a closed and bounded subset of $\mathbb{R}$, and so this set admits a maximum. We know that the set $\{\hat{T}\}$ is closed and bounded because the function that maps a position at the beginning of forward guidance to a threshold $\hat{T}$ is continuous.

### Appendix D. Model with government spending

We derive the Euler equation and Phillips curve in the model with government spending introduced in section 3.3—thus obtaining equations (22) and (21). We then linearize the dynamical system describing the equilibrium and study the dynamics of the model.

#### Derivation of Euler equation and Phillips curve

**Government spending.** We begin by computing the government’s spending on each good. At any time $t$ the government chooses the amounts $g_j(t)$ of each good $j \in [0, 1]$ to minimize the expenditure

$$\int_0^1 p_j(t)g_j(t) \, dj$$

subject to the constraint of providing an amount of public consumption $g$:

$$\left[ \int_0^1 g_j(t)^{(\epsilon-1)/\epsilon} \, dj \right]^{\epsilon/(\epsilon-1)} = g(t).$$

To solve the government’s problem at time $t$, we set up a Lagrangian:

$$\mathcal{L} = \int_0^1 p_j(t)g_j(t) \, dj + C \cdot \left\{ g - \left[ \int_0^1 g_j(t)^{(\epsilon-1)/\epsilon} \, dj \right]^{\epsilon/(\epsilon-1)} \right\},$$

where $C$ is the Lagrange multiplier on the public-consumption constraint. We then follow the same steps as in the derivation of (A4). The first-order conditions with respect to $g_j(t)$ for all $j \in [0, 1]$ are $\partial \mathcal{L}/\partial g_j = 0$. These conditions imply

$$(A25) \quad p_j(t) = C \cdot \left[ \frac{g_j(t)}{g(t)} \right]^{-1/\epsilon}.$$  

 Appropriately integrating (A25) over all $j \in [0, 1]$, and using (3) and (20), we find

$$(A26) \quad C = p(t).$$

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Combining (A25) and (A26), we obtain the government’s demand for good \( j \):

\[
g_j(t) = \left[ \frac{p_j(t)}{p(t)} \right]^{-\epsilon} g(t).
\]

**Household saving and pricing.** Next we determine household \( j \)’s saving and pricing. The current-value Hamiltonian of the household’s problem is

\[
\mathcal{H}_j = \frac{\epsilon}{\epsilon - 1} \ln \left( \int_0^1 c_{jk}(t)^{\frac{\epsilon - 1}{\epsilon}} \, dk \right) + u \left( \frac{b_j(t) - b(t)}{p(t)} \right) - \frac{1}{1 + \eta} \left[ \frac{\kappa}{a} y_j^d(p_j(t), t) \right]^{1+\eta} - \frac{Y}{2} \pi_j(t)^2 \\
+ \mathcal{A}_j(t) \left[ i(t) b_j(t) + p_j(t) y_j^d(p_j(t), t) - \int_0^1 p_k(t) c_{jk}(t) \, dk - \tau(t) \right] + \mathcal{B}_j(t) \pi_j(t) p_j(t).
\]

The Hamiltonian’s terms including the consumption levels \( c_{jk}(t) \) are the same as in appendix A, so the optimality conditions \( \partial \mathcal{H}_j / \partial c_{jk} = 0 \) remain the same, which implies that (A1), (A2), and (A3) remain valid.

Adding the government’s demand, given by (A27), to households’ demand, given by (A3), we obtain the total demand for good \( j \) at time \( t \):

\[
y_j^d(p_j(t), t) = g_j(t) + \int_0^1 c_{jk}(t) \, dk = \left[ \frac{p_j(t)}{p(t)} \right]^{-\epsilon} y(t),
\]

where \( y(t) \equiv g(t) + \int_0^1 c_j(t) \, dj \) measures total consumption. The expression for \( y_j^d(p_j(t), t) \) enters into the Hamiltonian \( \mathcal{H}_j \).

The Hamiltonian’s terms including bond holdings \( b_j(t) \) are the same as in appendix A; therefore, the optimality condition \( \partial \mathcal{H}_j / \partial b_j = \delta \mathcal{A}_j - \mathcal{B}_j \) remains the same, implying that the Euler equation (A5) remains valid.

The Hamiltonian’s terms including inflation \( \pi_j(t) \) are also the same as in appendix A, so the optimality condition \( \partial \mathcal{H}_j / \partial \pi_j = 0 \) is unchanged. Equations (A6) and (A7) therefore remain valid.

Last, because the disutility from labor is convex, the optimality condition \( \partial \mathcal{H}_j / \partial p_j = \delta \mathcal{B}_j - \mathcal{B}_j \) is modified. The condition now yields

\[
\frac{\epsilon}{p_j} \left( \frac{\kappa}{a} y_j^d \right)^{1+\eta} + (1 - \epsilon) \mathcal{A}_j y_j + \mathcal{B}_j \pi_j = \delta \mathcal{B}_j - \mathcal{B}_j,
\]

which can be rewritten

\[
\pi_j - \frac{(\epsilon - 1) y_j^d \mathcal{A}_j}{\mathcal{B}_j p_j} \left[ p_j - \frac{\epsilon}{\epsilon - 1} \left( \frac{\kappa}{a} y_j^d \right)^{1+\eta} \frac{\mathcal{A}_j}{\mathcal{A}_j} \right] = \delta - \frac{\mathcal{B}_j}{\mathcal{B}_j}.
\]
Then, using (A2), (A6), and (A7), we obtain the household’s Phillips curve:

\[
\frac{\pi_j}{\pi_j} = \delta + \frac{(\epsilon - 1)\gamma_j}{\gamma c_j \pi_j} \left[ \frac{p_j}{P} - \frac{\epsilon}{\epsilon - 1} \left( \frac{\kappa}{a} \right)^{1+\eta} y_j c_j \right].
\]

\[\text{(A28)}\]

**Equilibrium saving and pricing.** In equilibrium the household’s Euler equation, given by (A5), simplifies because all households behave the same, so \( c_j = c \) and \( b_j = b \). Accordingly the Euler equation reduces to

\[\frac{\dot{c}}{c} = r - \delta + u' \left( 0 \right) c.\]

This is just the Euler equation given by (21). Unlike in appendix A, production and consumption are not equal (since \( y = c + g \)); therefore, we can no longer use \( y \) instead of \( c \).

The household’s Phillips curve, given by (A28), also simplifies because all households behave the same, so \( p_j = p, \pi_j = \pi, y_j = y, \) and \( c_j = c \); and because output equals private plus public consumption, so \( y = c + g \). Hence the Phillips curve reduces to:

\[\dot{\pi} = \delta \pi + \frac{(\epsilon - 1)(c + g)}{\gamma c} \left[ 1 - \frac{\epsilon}{\epsilon - 1} \left( \frac{\kappa}{a} \right)^{1+\eta} (c + g)^\eta c \right].\]

This is just the Phillips curve given by (22).

**Linearization**

The convexity of the disutility of labor makes the Phillips curve (22) nonlinear, so additional steady states may appear in normal times and at the ZLB. We circumvent this issue as in the literature: by studying dynamics around the steady state closest to the natural steady state. To that end, we linearize the Euler equation and Phillips curve around the natural steady state, and we analyze the resulting linear dynamical system.

**Euler equation.** We begin by linearizing the Euler equation (21) around the point \([c = c^n, \pi = 0]\). We consider two different monetary-policy rules. First, when monetary policy is normal, \( r(\pi) = r^n + (\phi - 1)\pi \). Then the Euler equation is

\[\dot{c} = E(c, \pi),\]

where

\[E(c, \pi) = c \left[ (\phi - 1)\pi + u'(0)(c - c^n) \right].\]

The linearized version is

\[\dot{c} = E(c^n, 0) + \frac{\partial E}{\partial c}(c - c^n) + \frac{\partial E}{\partial \pi} \pi.\]
where the partial derivatives are evaluated at \([c = c^n, \pi = 0]\). We have \(E(c^n, 0) = 0\) and

\[
\frac{\partial E}{\partial c} = c^n u'(0), \quad \frac{\partial E}{\partial \pi} = c^n (\phi - 1).
\]

So the linearized Euler equation with normal monetary policy is

\[
(A29) \quad \dot{c} = c^n \left[ (\phi - 1)\pi + u'(0)(c - c^n) \right].
\]

Second, when monetary policy is at the ZLB, \(r(\pi) = -\pi\). Then the Euler equation can be written \(\dot{c} = E(c, \pi)\) where

\[
E(c, \pi) = c \left[ -r^n - \pi + u'(0)(c - c^n) \right].
\]

The linearized version is

\[
\dot{c} = E(c^n, 0) + \frac{\partial E}{\partial c}(c - c^n) + \frac{\partial E}{\partial \pi} \pi,
\]

where the partial derivatives are evaluated at \([c = c^n, \pi = 0]\). We have \(E(c^n, 0) = -c^n r^n\) and

\[
\frac{\partial E}{\partial c} = c^n u'(0), \quad \frac{\partial E}{\partial \pi} = -c^n.
\]

So the linearized Euler equation at the ZLB is

\[
\dot{c} = c^n \left[ -r^n - \pi + u'(0)(c - c^n) \right],
\]

which gives (24).

**Phillips curve.** Next we linearize the Phillips curve (22) around the point \([c = c^n, \pi = 0, g = 0]\), where \(c^n\) is defined by

\[
c^n = \left( \frac{\epsilon - 1}{\epsilon} \right)^{1/(1+\eta)} \frac{a}{\kappa}.
\]

The Phillips curve can be written \(\dot{\pi} = P(c, \pi, g)\) where

\[
P(c, \pi, g) = \delta \pi + \frac{(\epsilon - 1)(c + g)}{\gamma c} \left[ 1 - \frac{\epsilon}{\epsilon - 1} \left( \frac{\kappa}{a} \right)^{1+\eta} (c + g)^{\eta} c \right].
\]

The linearized version is

\[
\dot{\pi} = P(c^n, 0, 0) + \frac{\partial P}{\partial c}(c - c^n) + \frac{\partial P}{\partial \pi} \pi + \frac{\partial P}{\partial g} g,
\]
where the partial derivatives are evaluated at \([c = c^n, \pi = 0, g = 0]\). We have \(P(c^n, 0, 0) = 0\) and

\[
\begin{align*}
\frac{\partial P}{\partial c} &= -\frac{\epsilon}{\gamma} \left( \frac{\kappa}{a} \right)^{1+\eta} (1+\eta)(c^n)\eta = -(1+\eta) \frac{\epsilon \kappa}{\gamma a} \left( \frac{\epsilon - 1}{\epsilon} \right)^{\eta/(1+\eta)} \\
\frac{\partial P}{\partial \pi} &= \delta \\
\frac{\partial P}{\partial g} &= -\frac{\epsilon}{\gamma} \left( \frac{\kappa}{a} \right)^{1+\eta} \eta (c^n)\eta = -\frac{\epsilon \kappa}{\gamma a} \left( \frac{\epsilon - 1}{\epsilon} \right)^{\eta/(1+\eta)}
\end{align*}
\]

Hence, the linearized Phillips curve is

\[
\dot{\pi} = \delta \pi - \frac{\epsilon \kappa}{\gamma a} \left( \frac{\epsilon - 1}{\epsilon} \right)^{\eta/(1+\eta)} \left[ (1+\eta)(c - c^n) + \eta g \right],
\]

which yields (25).

**Dynamics**

We first study the dynamics of the NK and WUNK models in normal times: the natural rate of interest is positive, monetary policy follows \(i(\pi) = r^n + \phi \pi\), and government spending is zero. In the neighborhood of the point \([c = c^n, \pi = 0, g = 0]\), the Euler equation (21) and Phillips curve (22) are linearly approximated by (A29) and (25). Thus we obtain the following proposition:

**Proposition A1.** Consider the NK and WUNK models with convex disutility of labor \((\eta > 0)\) and no government spending \((g = 0)\). Assume that the economy is in normal times. Then the models admit a steady state where inflation is zero \((\pi = 0)\), consumption is at its natural level \((c = c^n)\), and the ZLB is not binding \((i = r^n > 0)\). Around this natural steady state, dynamics are governed by the linear dynamical system

\[
\begin{bmatrix}
\frac{\partial}{\partial t} \hat{c} \\
\frac{\partial}{\partial t} \hat{\pi}
\end{bmatrix} =
\begin{bmatrix}
\frac{u' (0) c^n}{(\phi - 1) c^n} \\
-(1+\eta) \frac{\epsilon \kappa}{\gamma a} (\frac{\epsilon - 1}{\epsilon})^{\eta/(1+\eta)}
\end{bmatrix}
\begin{bmatrix}
\frac{c - c^n}{\pi}
\end{bmatrix}.
\]

In the NK model, the dynamical system is a source when monetary policy is active \((\phi > 1)\) and a saddle when monetary policy is passive \((\phi < 1)\). In the WUNK model, the dynamical system is a source whether monetary policy is active or passive.

**Proof.** The dynamics of the NK and WUNK models in the neighborhood of the point \([c = c^n, \pi = 0, g = 0]\) are well approximated by the linear dynamical system composed of (25) and (A29). This system is just (A30), whose matrix we denote \(M\).
As $[c = c^n, \pi = 0]$ satisfies (A30) with $\dot{\pi} = 0$ and $\dot{c} = 0$, it is a steady state of the models. The nominal interest rate at $[c^n, 0]$ is given by $i = r^n + \phi \times 0 = r^n > 0$.

As in the proofs of propositions 1 and 2, we classify the linear dynamical system (A30) using the trace and determinant of $M$:

$$\text{tr}(M) = \delta + u'(0)c^n$$

$$\det(M) = \delta c^n \left[u'(0) + (\phi - 1)(1 + \eta)\frac{\epsilon k}{\delta y a} \left(\frac{\epsilon - 1}{\epsilon}\right)^{\eta/(1+\eta)}\right].$$

In the NK model, $u'(0) = 0$ so the sign of $\det(M)$ is given by the sign of $\phi - 1$. Accordingly, when $\phi > 1$, $\det(M) > 0$; since $\text{tr}(M) = \delta > 0$, the system (A30) is a source. In contrast, when $\phi < 1$, $\det(M) < 0$, indicating that the system (A30) is a saddle.

In the WUNK model, since $\phi - 1 \geq -1$ for any $\phi \geq 0$, we have

$$\det(M) \geq \delta c^n \left[u'(0) - (1 + \eta)\frac{\epsilon k}{\delta y a} \left(\frac{\epsilon - 1}{\epsilon}\right)^{\eta/(1+\eta)}\right].$$

Moreover, the WUNK assumption (23) says that the term in square brackets is positive, so $\det(M) > 0$. Since we also have $\text{tr}(M) > \delta > 0$, we conclude that the system (A30) is a source.

Proposition A1 extends proposition 1 to models with convex disutility of labor. It shows that in normal times, the dynamics of the models around the natural steady state remain the same.

We turn to the dynamics of the models at the ZLB: the natural rate of interest is negative, and monetary policy simply sets $i = 0$. In the neighborhood of the point $[c = c^n, \pi = 0, g = 0]$, the Euler equation (21) and Phillips curve (22) are linearly approximated by (24) and (25), respectively. Hence, we obtain the following results:

**Proposition A2.** Consider the NK and WUNK models with convex disutility of labor ($\eta > 0$) and government spending ($g \geq 0$). Assume that the economy is at the ZLB. Then the models admit a steady state where private consumption and inflation are given by

\begin{align*}
(A31) \quad c^g &= c^n + \frac{r^n + \frac{ek}{\delta y a} \left(\frac{\epsilon - 1}{\epsilon}\right)^{\eta/(1+\eta)} \eta g}{u'(0) - (1 + \eta)\frac{ek}{\delta y a} \left(\frac{\epsilon - 1}{\epsilon}\right)^{\eta/(1+\eta)}} \\
(A32) \quad \pi^g &= \frac{(1 + \eta)r^n + u'(0)\eta g}{u'(0)\frac{\delta y a}{ek} \left(\frac{\epsilon - 1}{\epsilon}\right)^{\eta/(1+\eta)} - (1 + \eta)}.
\end{align*}
Around the steady state, dynamics are governed by the linear dynamical system

\[
\begin{bmatrix}
\dot{c} \\
\dot{\pi}
\end{bmatrix} =
\begin{bmatrix}
u'(0)c^n & -c^n \\
-(1+\eta)\frac{\varepsilon k}{\gamma a} (\frac{\varepsilon - 1}{\varepsilon})^{\eta/(1+\eta)} & \delta
\end{bmatrix}
\begin{bmatrix}
c - c^g \\
\pi - \pi^g
\end{bmatrix}.
\]

In the NK model, the dynamical system is a saddle. In the WUNK model, the dynamical system is a source.

**Proof.** The dynamics of the NK and WUNK models around the point \([c = c^n, \pi = 0, g = 0]\) are well approximated by the linear dynamical system composed of (24) and (25).

A steady state needs to satisfy (24) and (25) with \(\dot{\pi} = 0\) and \(\dot{c} = 0\). These equations form the linear system

\[
\begin{align*}
\pi &= \varepsilon k \frac{\varepsilon - 1}{\delta \gamma a} \left(\frac{\varepsilon - 1}{\varepsilon}\right)^{\eta/(1+\eta)} [1 + \eta](c - c^n) + \eta g \\
\pi &= -r^n + u'(0)(c - c^n).
\end{align*}
\]

A solution to this system with positive consumption is a steady state.

In the NK model, \(u'(0) = 0\), so the system admits a unique solution:

\[
\begin{align*}
(A34) & \quad c = c^n - \frac{\eta}{1 + \eta} g - \frac{\delta \gamma a}{\varepsilon k} \cdot \frac{1}{1 + \eta} \left(\frac{\varepsilon}{\varepsilon - 1}\right)^{\eta/(1+\eta)} r^n \\
(A35) & \quad \pi = -r^n.
\end{align*}
\]

Since \(r^n < 0\), for \(g\) not too large, the solution has positive consumption so it is a steady state. Hence the NK model admits a unique steady state at the ZLB. Note that (A31) and (A32) reduce to (A34) and (A35) when \(u'(0) = 0\).

In the WUNK model, since (23) holds, the two equations in the linear system are non-parallel, so the system admits a unique solution. Inflation and consumption in that unique solution are given by (A32) and (A31). Whether inflation is positive or negative, and whether consumption is above or below its natural level, depend on the amount of government spending. Furthermore, for \(g\) and \(r^n\) close enough to zero, \(c^g\) is close enough to \(c^n\) and positive. Then the solution to the system has positive consumption, implying that it is indeed a steady state.

The linear dynamical system (A33) is just the system composed of (25) and (A29), rewritten in canonical form. Hence (A33) indeed governs the dynamics of the NK and WUNK models. We classify the system (A33), whose matrix we denote \(M\), using the same methodology. The trace
and determinant of $M$ are

$$
\text{tr}(M) = \delta + u'(0)c^n
$$

$$
\det(M) = \delta c^n \left[ u'(0) - (1 + \eta) \frac{\epsilon K}{\delta \alpha} \left( \frac{\epsilon - 1}{\epsilon} \right)^{\eta/(1+\eta)} \right].
$$

In the NK model, $u'(0) = 0$ so $\det(M) < 0$, indicating that the system (A33) is a saddle. In the WUNK model, since we assume (23), $\det(M) > 0$. Since we also have $\text{tr}(M) > 0$, we conclude that the system (A33) is a source. Using the same argument as at the end of the proof of proposition 2, we can also show that the system is a nodal source, not a spiral source.

Proposition A2 extends proposition 2 to models with convex disutility of labor and government spending. It shows that at the ZLB, the dynamics of the models remain the same.

### Appendix E. Proofs for the model with government spending

We complement the graphical proofs of propositions 5 and 9 developed in the main text. The propositions pertain to the model with government spending.

#### Proof of proposition 5: complement

We characterize the amount $g^*$ in the NK model, and we compute the limit of the government-spending multiplier in the WUNK model.

In the NK model, the amount $g^*$ of government spending is the amount that makes the unstable line of the dynamical system go through the natural steady state. With a bit less spending than $g^*$ (panel B of figure 6), the natural steady state is below the unstable line and is connected to trajectories coming from the southwest quadrant of the phase diagram. Hence, for $g < g^*$, $\lim_{T \to \infty} c(0; g) = -\infty$. With a bit more spending than $g^*$ (panel D of figure 6), the natural steady state is above the unstable line and is connected to trajectories coming from the northeast quadrant of the phase diagram. Hence, for $g > g^*$, $\lim_{T \to \infty} c(0; g) = +\infty$. Accordingly, for any $s > 0$, $\lim_{T \to \infty} m(g^*, s) = +\infty$.

In the WUNK model, when the ZLB is infinitely long-lasting, the economy jumps to the ZLB steady state at time 0: $\lim_{T \to \infty} c(0; g) = c^g(g)$, where $c^g(g)$ is given by (A31). The steady-state consumption $c^g(g)$ is linear in government spending $g$, with a coefficient in front of $g$ of

$$
\frac{\eta}{u'(0) \frac{\delta \alpha}{\epsilon K} \left( \frac{\epsilon - 1}{\epsilon} \right)^{\eta/(1+\eta)} - (1 + \eta)}.
$$
Accordingly, for any $s > 0$, we have

\[
\lim_{T \to \infty} m(g, s) = 1 + \frac{\lim_{T \to \infty} c(0; g - s/2) - \lim_{T \to \infty} c(0; g + s/2)}{s} \\
= 1 + \frac{c^d(g + s/2) - c^d(g - s/2)}{s} \\
= 1 + \frac{\eta}{u'(0) \frac{\delta y}{\epsilon k} \left( \frac{\epsilon}{\epsilon - 1} \right)^{\eta/(1+\eta)} - (1 + \eta)},
\]

which yields (28).

**Proof of proposition 9: complement**

We compute the government-spending multiplier at the ZLB in the WUNK model. Private consumption and inflation at the ZLB steady state are determined by (A31) and (A32). The coefficients in front of government spending $g$ in these expressions are

\[
\frac{\eta}{u'(0) \frac{\delta y}{\epsilon k} \left( \frac{\epsilon}{\epsilon - 1} \right)^{\eta/(1+\eta)} - (1 + \eta)} \quad \text{and} \quad \frac{u'(0)\eta}{u'(0) \frac{\delta y}{\epsilon k} \left( \frac{\epsilon}{\epsilon - 1} \right)^{\eta/(1+\eta)} - (1 + \eta)}.
\]

Since (23) holds, both coefficients are positive. Hence, an increase in $g$ raises private consumption and inflation. Moreover, $dc/dg$ is given by the first of these coefficient; this immediately gives us the expression for the multiplier $dy/dg = 1 + dc/dg$. 

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