

# **Problem Set on Optimal Control**

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### Problem 1.

Consider the following optimal growth problem: Given initial capital  $k_0 > 0$ , choose a consumption path  $\{c_t\}_{t \geq 0}$  to maximize utility

$$\int_0^{\infty} e^{-\rho \cdot t} \cdot \ln(c_t) dt$$

subject to the law of motion of capital

$$\dot{k}_t = f(k_t) - c_t - \delta \cdot k_t.$$

The discount factor  $\rho > 0$ , and the production function  $f$  satisfies

$$f(k) = A \cdot k^\alpha,$$

where  $\alpha \in (0, 1)$  and  $A > 0$ .

- A. Write down the present-value Hamiltonian.
- B. Show that the Euler equation is

$$\frac{\dot{c}_t}{c_t} = \alpha \cdot A \cdot k_t^{\alpha-1} - (\delta + \rho).$$

- C. Solve for the steady state of the system.

## Problem 2.

Consider the following investment problem: Given initial capital  $k_0$ , choose the investment path  $\{i_t\}_{t \geq 0}$  to maximize profits

$$\int_0^{\infty} e^{-r \cdot t} \left[ f(k_t) - i_t - \frac{\chi}{2} \cdot \left( \frac{i_t^2}{k_t} \right) \right] dt$$

subject to the law of motion of capital (we assume no capital depreciation)

$$\dot{k}_t = i_t.$$

The interest rate  $r > 0$ , the capital adjustment cost  $\chi > 0$ , and the production function  $f$  satisfies  $f' > 0$  and  $f'' < 0$ .

- A. Write down the current-value Hamiltonian.
- B. Use the optimality conditions for the current-value Hamiltonian to derive the following differential equations:

$$\begin{aligned} \dot{k}_t &= \left( \frac{q_t - 1}{\chi} \right) \cdot k_t \\ \dot{q}_t &= r \cdot q_t - f'(k_t) - \frac{1}{2 \cdot \chi} (q_t - 1)^2 \end{aligned}$$

- C. Solve for the steady state.